Local moment maps and the splitting of classical multiplets

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February 7, 2008

Abstract

We generalize the concept of global moment maps to local moment maps whose different branches are labelled by the elements of the fundamental group of the underlying symplectic manifold. These branches can be smoothly glued together by employing fundamental-group-valued Čech cocycles on the phase space. In the course of this work we prove a couple of theorems on the liftability of group actions to symplectic covering spaces, and examine the possible extensions of the original group by the fundamental group of the quotient phase space. It it shown how the splitting of multiplets, this being a consequence of the multiply-connectedness of the quotient phase space, can be described by identification maps on a space of multiplets derived from a symplectic universal covering manifold. The states that are identified in this process are related by certain integrals over non-contractible loops in the quotient phase spacec.

Contents

1 Introduction

For a physical system whose Lagrangian is invariant under the action of the isometry group G of a background spacetime, the algebra of conserved charges coincides with the Noether charge algebra of the system associated with the isometry group. In recent years, it was observed by many authors working in the field of high energy physics, that when classical or quantum fields propagate in background spacetimes which are topologically non-trivial then the associated algebra of conserved charges may reflect this non-triviality by exhibiting extensions of the Noether charge algebra which measure the topology of the background, provided the system is only semi-invariant under the isometry group of the spacetime (see, e.g., [1]). On the other hand, the algebra of conserved charges defines a partition of the underlying phase space by distinguishing those subsets of the phase space on which the values of all conserved charges

involved are constant (this is just the first step to a Marsden-Weinstein reduction). These subsets then could be called "elements of classical G-multiplets", a multiplet being defined as a G-orbit of such subsets. On every such multiplet, the symmetry group G acts transitively, thus exhibiting a property which is the classical analogue of an irreducible representation in the quantum theory [2]. Now, if dynamical systems are described in the framework of symplectic formalism, the quantity determining this phase space partition is what we call a global moment map on the phase space. However, this global moment map need not exist in the case that the phase space is not simply connected. This observation was the starting point for the investigations in this work. For systems with finite degrees of freedom, we generalize the concept of a global moment map to multiply connected phase spaces and general symplectic manifolds. We show that the appropriate generalization involves a locally defined multi-valued moment map, whose different branches are labelled by the fundamental group of the phase space. Furthermore, it is shown how the different local branches can be smoothly glued together by a glueing condition, which is expressed by certain Čech cocycles on the underlying symplectic manifold. These constructions are intimately related with the existence of a universal symplectic covering manifold of the original phase space. On the covering, global moment maps defining G-multiplets always exist. At the end of this work we show how these multiplets on the covering space can be related to G-multiplets on the original symplectic manifold by an identification map which derives from the covering projection.

In order to formulate these ideas rigourously we first had to examine the question of liftability of symplectic group actions on a symplectic manifold to a covering space. In the course of this, we proved a series of theorems investigating the existence and uniqueness of such lifts, and to which extent the lifted action preserves the group structure of the original symmetry group G.

The plan of the paper is as follows: In section 2 we collect basic statements about covering spaces on which the rest of this work relies. In section 3 and 4 we examine multi-valued potential functions for closed but not exact differential forms on a multiply connected manifold. These considerations will be needed later on when examining local moment maps. In section 5–7 we collect notation conventions and basic facts about symplectic manifolds, Hamiltonian vector fields, and cotangent bundles. In section 8 we study how covering projections can be extended to local symplectomorphisms of coverings of cotangent bundles. Sections 9 and 10 examine the conditions under which an action of a Lie group G on a manifold can be lifted to an action on a covering manifold, and when such a lift preserves the group law of G. Section 11 examines the relation between group actions on covering spaces and equivariance of the covering map. In 12 we introduce our notation for symplectic G-actions on a symplectic manifold. Section 13 recapitulates the notion of global moment maps as they are usually defined, while this concept is generalized to local moment maps in the subsequent section 14. Equivariance of global and local moment maps is discussed in section 15. Section 16 discusses the relation of local moment maps to covering spaces which are themselves multiply connected. In 17 we introduce the concept of G-states, while the subsequent section 18 shows how a splitting

of G-states on multiply connected symplectic manifold arises.

2 Basic facts about coverings

In this section we quote the main results on coverings and lifting theorems on which the rest of this work is built; it is mainly based on [3],[4],[5],[6]. In this work we are interested in covering spaces that are manifolds. Consequently, some of the definitions and quotations of theorems to follow are not presented in their full generality, as appropriate for general topological spaces, but rather we give working definitions and formulations pertaining to manifolds and the fact that these are special topological spaces (locally homeomorphic to $\mathbb{R}^n$).

— A covering of manifolds is a triple (p, X, Y), where $p : X \to Y$ is a smooth surjective map of smooth manifolds X, Y, where Y is connected, such that for every $y \in Y$ there exists an open neighbourhood $V \subset Y$ of y for which $p^{-1}(V)$ is a disjoint union of open sets U in X, on each of which the restriction $p|U : U \to Y$ is a diffeomorphism. Every open $V \subset Y$ for which this is true is called admissible (with respect to p). This means that, for each $y \in Y$, the inverse image $p^{-1}(y) \subset X$, called the fibre over y, is discrete. Since Y is connected, all fibres have the same cardinality. p is called projection or covering map. A diffeomorphism $\phi : X \to X$ such that $p \circ \phi = p$ is called a deck transformation of the covering. The set $\mathcal{D}$ of all deck transformations of the covering is a group under composition of maps. Since every $\phi \in \mathcal{D}$ permutes the elements in the fibres $p^{-1}(y)$, the group $\mathcal{D}$ of all ϕ is discrete. If X is connected, deck transformations are uniquely determined by their value at a given point $x \in X$.

— Let $x \in p^{-1}(y)$, and let $\pi_1(X, x)$, $\pi_1(Y, y)$ denote the fundamental groups of X and Y based at x and y, respectively. The projection p induces a homomorphism $p_\# : \pi_1(X, x) \to \pi_1(Y, y)$ of fundamental groups, such that $p_\# \pi_1(X, x')$ ranges through the set of all conjugates of $p_\# \pi_1(X, x)$ in $\pi_1(Y, y)$, as x' ranges through the elements in the fibre $p^{-1}(y)$. A covering $p : X \to Y$ is called normal if $p_\# \pi_1(X, x')$ is normal in $\pi_1(Y, y)$ for some (hence any) $x' \in p^{-1}(y)$. One can show that a covering is normal if and only if the deck transformation group $\mathcal{D}$ acts transitively on the fibres, i.e. for all $x, x' \in X$ with $p(x) = p(x')$ there exists a unique deck transformation $\phi \in \mathcal{D}$ with $x' = \phi(x)$. This is certainly true when X is simply connected; in this case, the deck transformation group $\mathcal{D}$ is isomorphic to the fundamental group $\pi_1(Y, y)$.

— Let Γ be a group of diffeomorphisms acting on the manifold X, and let Γx denote the orbit of x under Γ. The set of all orbits is denoted by X/Γ, and is called an orbit space. The natural projection $pr : X \to X/\Gamma$ sends each $x \in X$ to its orbit, $pr(x) = \Gamma x$. The topology on X/Γ is the quotient topology, for which p is continuous, and an open map. Γ acts properly discontinuously on X if every $x \in X$ has a neighbourhood U such that the set $\{\gamma \in \Gamma \mid \gamma U \cap U \neq \emptyset\}$ is finite. Γ acts freely if no $\gamma \neq e$ has a fixed point in x. Γ acts properly discontinuously and freely if each $x \in X$ has a neighbourhood U such that $e \neq \gamma \in \Gamma$ implies $\gamma U \cap U = \emptyset$.

— Given a covering $p : X \to Y$ and a map $f : V \to Y$ defined on some manifold V, a map $\tilde{f} : V \to X$ is called a lift of f through p if $p \circ \tilde{f} = f$. Two maps $f, g : V \to Y$ are called homotopic if there exists a continuous map $G : I \times V \to Y$, $(t, v) \mapsto G_t(v)$, such that $G_0 = f$ and $G_1 = g$. G is called a homotopy of f and g.

— We now quote without proof a couple of theorems from [6] which we will make us of frequently. These theorems are actually proven in every textbook on Algebraic Topology.

Theorem: If Γ acts properly discontinuously and freely on the connected manifold X, the natural projection $pr : X \to X/\Gamma$ onto the orbit space is a covering map, and the covering is normal. Furthermore, the deck transformation group $\mathcal{D}$ of this covering is Γ.

As a converse, we have

Theorem: If $p : X \to Y$ is a covering and $\mathcal{D}$ is the group of deck transformations, then $\mathcal{D}$ acts properly discontinuously and freely on X. — If X is a simply connected manifold, every covering $p : X \to Y$ is a natural projection $pr : X \to X/\Gamma$ for some discrete group Γ of diffeomorphisms acting properly discontinuously and freely on X.

As a consequence, we have

Corollary: Let Y be a connected manifold. Then Y is diffeomorphic to an orbit space X/Γ, where X is a simply connected covering manifold X, and Γ is a group of diffeomorphisms acting properly discontinuously and freely on X. In this case, $\Gamma = \mathcal{D}$ coincides with the deck transformation group of the covering.

— Existence and uniqueness of lifts are determined by the following lifting theorems:

Theorem ("Unique Lifting Theorem"): Let $p : X \to Y$ be a covering of manifolds. If V is a connected manifold, $f : V \to Y$ is continuous, and g_1 and g_2 are lifts of f through p that coincide in one point, $g_1(v) = g_2(v)$, then $g_1 = g_2$.

Theorem ("Covering Homotopy Theorem"): Let $p : X \to Y$ be a covering. Let $f, g : V \to Y$ be continuous, let $\tilde{f} : V \to X$ be a lift of f, and let G be a homotopy of f and g. Then there is a unique pair $\left(\tilde{G}, \tilde{g}\right)$, where $\tilde{g}$ is a lift of g, $\tilde{G}$ is a lift of $\tilde{G}$, and $\tilde{G}$ is a homotopy of $\tilde{f}$ and $\tilde{g}$.

Theorem ("Lifting Map Theorem"): Let $p : X \to Y$ be a covering, let V be a connected manifold, let $f : V \to Y$ be continuous. Let $x \in X$, $y \in Y$, $v \in V$ such that $p(x) = y = f(v)$. Then f has a lift $\tilde{f} : V \to X$ with $\tilde{f}(v) = x$ if and only if $f_{\#}\pi_1(V, v) \subset p_{\#}\pi_1(X, x)$, where $f_{\#}$ denotes the homomorphism of fundamental groups induced by f.

3 Covering spaces and differential forms

Let $p : X \to Y$ be a covering of smooth manifolds. In this case the covering projection p is a local diffeomorphism. In this section we examine the relation between differential forms on X and Y which are related by pull-back through the covering map p, and where X is simply connected. The results will be needed to formulate the concept of local moment maps on non-simply connected symplectic manifolds in section 14.

3.1 Differential forms on X and Y

First consider a smooth q-form ω on Y. The pull-back $\Omega \equiv p^*\omega$ is a q-form on X, and Ω is invariant under the deck transformation group of the covering: For, let $\gamma \in \mathcal{D}$, then $\gamma^*\Omega = (p \circ \gamma)^* \omega = \Omega$. Conversely, let Ω be a q-form on X. Then, locally, Ω can be pulled back to Y to give a multi-valued q-form on Y, since the covering map p is a local diffeomorphism: To see this, let V be an admissible neighbourhood of a point $y \in Y$, such that the inverse image $p^{-1}(V)$ is a disjoint union of neighbourhoods U_i in X. On each U_i, the restriction $p|\,U_i : U_i \to Y$ is a diffeomorphism, so that we can pull back $\Omega|\,U_i \mapsto (p|\,U_i)^{-1*}\,\Omega$ on V. This gives a multi-valued q-form on V, each branch being labelled by some i. We ask under which condition all branches coincide. If this happens to be, we have $(p|\,U_i)^{-1*}\,\Omega = (p|\,U_j)^{-1*}\,\Omega$ for all i, j labelling different neighbourhoods U_i, U_j; but this means that

$$\left[(p|\,U_i)^{-1} \circ (p|\,U_j)\right]^* (\Omega|\,U_i) = (\Omega|\,U_j) \quad . \tag{1}$$

For a general covering we can proceed no further, since there need not exist a deck transformation mapping the neighbourhoods U_i and U_j into each other. Such a deck transformation exists, however, if X is simply connected. If $x_i \in U_i$ and $x_j \in U_j$ such that $p(x_j) = p(x_i) = y$, then there is a unique deck transformation γ with $x_j = \gamma(x_i)$. On the other hand, the map on the LHS of (1) satisfies

$$(p|\,U_i) \circ \left[(p|\,U_i)^{-1} \circ (p|\,U_j)\right] = (p|\,U_j) \tag{2}$$

and maps x_j to x_i; hence it coincides with the restriction $(\gamma \mid U_j, U_i)$. This argument can be performed for any neighbourhood U of some point $x \in X$; in turn, this implies that $\gamma^*\Omega = \Omega$. The fact that $\mathcal{D}$ now acts transitively on the fibres of the covering means that this relation must hold for all deck transformations $\gamma \in \mathcal{D}$.

As a consequence, if a q-form Ω on X is invariant under $\mathcal{D}$, then there is a uniquely determined q-form ω on Y such that $\Omega = p^*\omega$; for, ω is defined by mapping any vector $W \in T_yY$ to $(p_{loc})_*\,W$, where p_{loc} denotes the restriction of p to one of the connected components of the inverse image of an admissible neighbourhood of y in Y, and subsequently performing the pairing $\langle \Omega, (p_{loc})_*\,W \rangle$. Because of the $\mathcal{D}$-invariance of Ω, it does not matter which connected component we choose, and hence this construction is well-defined.

Altogether, we have shown

3.2 Proposition

Let $p : X \to Y$ be a covering of connected manifolds, where X is simply connected. Then a smooth q-form Ω on X is the pull-back of a q-form ω on Y, $\Omega = p^*\omega$, if and only if $\gamma^*\Omega = \Omega$ for all $\gamma \in \mathcal{D}$.

We now discuss the closure and exactness of forms on X and Y that are related by the covering map according to $\Omega = p^*\omega$. Since closure of a differential form is a local property, Ω is closed if and only if ω is closed. For if $d\omega = 0$, then $dp^*\omega = p^*d\omega = 0$; and conversely, if V is an admissible neighbourhood in Y, $d\left(\omega|\, V\right) = d\left[\left(p|\, U_i\right)^{-1*}\Omega\right] = 0$, if $d\Omega = 0$; here U_i is any neighbourhood in X that projects down to V. This result is actually independent of whether X is simply connected or not, and makes use only of the existence of a form Ω such that $\Omega = p^*\omega$.

Now we examine exactness. Trivially, if the form ω on Y is exact, then Ω is exact, since then $\Omega = p^*d\alpha = d\left(p^*\alpha\right)$. The converse is not true in general, however. For assume that $\Omega = d\eta$ for a $(q-1)$-form η on X. Assuming that $\Omega = p^*\omega$, proposition 3.2 says that $d\left(\gamma^*\eta - \eta\right) = 0$. Thus $\gamma^*\eta - \eta$ is closed, and since X is simply connected, it is also exact. This means that there exists a $(q-2)$-form $\chi\left(\gamma\right)$ on X such that

$$\gamma^*\eta = \eta + d\chi\left(\gamma\right) \quad . \tag{3}$$

Now unless $d\chi = 0$, we see from proposition 3.2 that η can**not** be the pull-back of a $(q-1)$-form on Y under p, as this requires η to be $\mathcal{D}$-invariant. Thus, although Ω is exact, ω need not be exact; it is closed, however, and hence defines an element $[\omega] \in H^q_{deRham}\left(Y\right)$.

The content of the last two paragraphs can be cast into a convenient form by introducing *$\mathcal{D}$-invariant cohomology classes* of forms on X:

3.3 Definition

Let $\Lambda^q_{\mathcal{D}I}\left(X\right)$ denote the subspace of all $\mathcal{D}$-invariant q-forms on X, i.e. $\gamma^*\Omega = \Omega$ for all $\gamma \in \mathcal{D}$. Let $Z^q_{\mathcal{D}I}\left(X\right)$ denote the class of all elements Ω in $\Lambda^q_{\mathcal{D}I}\left(X\right)$ which are closed under d, $d\Omega = 0$. Let $B^q_{\mathcal{D}I}\left(X\right)$ denote the class of forms Ω in $\Lambda^q_{\mathcal{D}I}\left(X\right)$ which are exact under d, i.e. there exists a $\eta \in \Lambda^{q-1}_{\mathcal{D}\mathcal{I}}\left(X\right)$ such that $\Omega = d\eta$. Now define the q-th *$\mathcal{D}$-invariant cohomology group* $H^q_{\mathcal{D}I}\left(X\right)$ on X as the quotient

$$H^q_{\mathcal{D}I}\left(X\right) \equiv Z^q_{\mathcal{D}I}\left(X\right)/B^q_{\mathcal{D}I}\left(X\right) \quad . \tag{4}$$

Formula (3) shows that $B^q_{\mathcal{D}I}\left(X\right) \underset{\neq}{\subset} Z^q_{\mathcal{D}I}\left(X\right)$ in general, and so $\dim H^q_{\mathcal{D}I}\left(X\right)$ can be non-vanishing although X has trivial de Rham cohomology groups. This is expressed in the next proposition, which is a consequence of proposition 3.2 and the discussion in the last paragraph:

3.4 Proposition

Assume that X is simply connected. Then the pull-back $p^*\Lambda^q(Y)$ of the space of q-forms $\Lambda^q(Y)$ on Y by the covering map p coincides with the subspace of all $\mathcal{D}$-invariant q-forms $\Lambda^q_{\mathcal{D}I}(X)$ on X, and p^* is a group isomorphism onto $\Lambda^q_{\mathcal{D}I}(X)$. Furthermore, $Z^q_{\mathcal{D}I}(X) = p^*Z^q(Y)$ and $B^q_{\mathcal{D}I}(X) = p^*B^q(Y)$, and therefore

$$\dim H^q_{\mathcal{D}I}(X) = \dim \frac{p^*Z^q(Y)}{p^*B^q(Y)} = \dim \frac{Z^q(Y)}{B^q(Y)} = \dim H^q(Y) \ > \ 0 \qquad (5)$$

in general.

There is a cohomological description of formula (3) in terms of special $\Lambda^*(X)$-valued cochains, where $\Lambda^*(X)$ denotes the ring of differential forms on X; the associated cohomology is defined and described in the appendix, section B. For the work pursued here the general case has no immediate application, but the case when X is simply connected and the forms involved are 1-forms is important. To start, let α be a closed 1-form on Y; then $p^*\alpha$ is closed on X, hence exact, since X is simply connected. Thus there exists a smooth function $F : X \to \mathbb{R}$ with $dF = p^*\alpha$. The discussion in proposition 3.2 has shown that dF is $\mathcal{D}$-invariant; therefore, $d(\gamma^*F - F) = 0$, or

$$F \circ \gamma - F \equiv c(\gamma) \in \mathbb{R} \qquad (6)$$

is a real constant on X, depending only on γ. In particular, $c(\gamma) \circ \gamma' = c(\gamma)$, and it follows that

$$c(\gamma\gamma') = F \circ (\gamma\gamma') - F = [F + c(\gamma)] \circ \gamma' - F = c(\gamma) + c(\gamma') \quad ; \qquad (7)$$

hence $c : \mathcal{D} \to \mathbb{R}$, $\gamma \mapsto c(\gamma)$ is a real 1-dimensional representation of $\mathcal{D}$. We have proven:

3.5 Proposition

Let $p : X \to Y$ be a covering of smooth manifolds, where X is simply connected. Let α be a closed 1-form on Y. Then the pull-back $p^*\alpha$ is an exact γ-invariant 1-form on X, with $p^*\alpha = dF$, where $F \in \mathcal{F}(X)$ is a smooth function on X. Under $\mathcal{D}$-transformations, F is invariant up to a real 1-dimensional $\mathcal{D}$-representation $c : \mathcal{D} \to \mathbb{R}$, i.e.

$$F \circ \gamma = F + c(\gamma) \quad . \qquad (8)$$

Now we see how the function F gives rise to multi-valued locally defined functions f_γ on Y, which represent local potentials, i.e. 0-forms, for the closed 1-form α: Given an admissible open neighbourhood $V \subset Y$, choose a connected component $U \subset X$ of $p^{-1}(V)$; then $p|U$ is a diffeomorphism onto V, and every other connected component in $p^{-1}(V)$ is obtained as the image of U under a

deck transformation γ. Since the sets γU, $\gamma \in \mathcal{D}$, are disjoint, this determines a collection (f_γ) of local potentials for α on V, each f_γ being defined as

$$f_\gamma = F \circ (p|\,\gamma U)^{-1} \quad . \tag{9}$$

By construction, we have $df_\gamma = \alpha$ for every $\gamma \in \mathcal{D}$. Furthermore, since $(p|\,\gamma U) \circ \gamma = p|\,U$, it follows that

$$f_\gamma = F \circ \left[\gamma \circ (p|\,U)^{-1}\right] = [F \circ \gamma] \circ (p|\,U)^{-1} = [F + c(\gamma)] \circ (p|\,U)^{-1} = f_e + c(\gamma) \quad , \tag{10}$$

where we have used (8).

We now prove the important result, that $c(\gamma)$ can be expressed as an integral of α over certain 1-cycles or loops in Y: To this end we recall that the deck transformation group $\mathcal{D}$ acts properly discontinuously and freely on the simply connected covering manifold X, and that Y is the orbit space $X/\mathcal{D}$. Furthermore, if x, y are base points of X, Y, with $p(x) = y$, then the fundamental group $\pi_1(Y, y)$ of Y at y is isomorphic to $\mathcal{D}$. This isomorphism is defined as follows: If $\gamma \in \mathcal{D}$, let λ be an arbitrary path in X connecting the base point x with its image $\gamma(x)$; then λ projects into a loop $p \circ \lambda$ at y, whose associated homotopy class $[p \circ \lambda]$ represents $\gamma \in \pi_1(Y, y)$. Any other choice λ' of path is homotopic to λ due to X being simply connected, hence the loops $[p \circ \lambda]$ and $[p \circ \lambda']$ both represent the same homotopy class. Conversely, given a loop l at y representing γ, there exists a unique lift $\tilde{l}$ of l through p with initial point $x \in p^{-1}(y)$ (as follows from theorem 2); this means that $p \circ \tilde{l} = l$, and $\tilde{l}(1) = \gamma(x)$. Now we can prove

3.6 Theorem

Let $y \in Y$ be the base point of Y, let l be a loop at y with $[l] = \gamma \in \pi_1(Y, y) \simeq \mathcal{D}$. Then

$$c(\gamma) = \int_l \alpha \quad . \tag{11}$$

The integral depends only on the homotopy class $[l]$ of l.

Proof :

The lift $\tilde{l}$ of l to the base point x of X satisfies $\tilde{l}(1) = \gamma(x)$. Since $p \circ \tilde{l} = l$, we have

$$\int_l \alpha = \int_{p\circ\tilde{l}} \alpha = \int_{\tilde{l}} p^*\alpha \quad ,$$

but $p^*\alpha = dF$, and since $\tilde{l}$ connects x and $\gamma(x)$, the last integral in the above equation is

$$\int_{\tilde{l}} p^*\alpha = \int_x^{\gamma(x)} dF = F \circ \gamma(x) - F(x) = [F \circ \gamma - F](x) = c(\gamma) \quad ,$$

according to the definition (8) of $c(\gamma)$. This proves (11). Any other loop l' homotopic to l lifts to a path $\tilde{l}'$ homotopic to $\tilde{l}$; hence the difference between the associated integrals is an integral of dF over a loop, which must vanish, as dF is exact. ■

3.7 Corollary

Any element γ of $\mathcal{D}$ that has torsion lies in the kernel of c; in other words, if there is a $k \in \mathbb{N}$ with $\gamma^k = e$ then $c(\gamma) = 0$.

Proof :

Insert $e = \gamma^k$ into $c(e) = 0$, which gives $0 = c\left(\gamma^k\right) = k \cdot c(\gamma)$ due to (7). Since $k > 0$, $c(\gamma) = 0$. ■

4 Čech cohomology and multi-valued functions

We now want to make more precise the notion of multi-valued functions that serve as local potentials for closed 1-forms on the non-simply connected manifold Y. We want to express the local potentials, as they were defined in formula (9), and their mutual relations, in terms of locally defined quantities and glueing conditions without explicit reference to a specific covering manifold. It is clear that we must make up for the information that is lost by discarding the covering space from consideration by some additional structure on the manifold Y. It turns out that the necessary ingredients are

1.) a countable simply connected path-connected open cover $\mathcal{V} = \{V_a \subset Y \mid a \in A\}$ of Y, i.e. a collection of countably many open sets $V_a \subset Y$ whose union gives Y, and such that every loop in V_a is homotopic in Y to a constant loop, and all V_a are path-connected.

Since Y is a manifold, a cover of the type just described always exists. We note that, as every element $V_a \in \mathcal{V}$ is simply connected (in Y), it is automatically admissible with respect to the covering map $p : X \to Y$; for, assume it were not admissible; then the inverse image $p^{-1}(V_a)$ contained a connected, hence path-connected, component U on which the restriction of p is not injective. In particular, there are points $x, x' \in U$, $x \neq x'$, but $p(x) = p(x')$. Choose a path λ connecting x and x' in U; then this projects into a loop in V_a at $p(x)$, which is non-contractible in Y. This contradicts the assumption of V_a being simply connected. — The second ingredient is

2.) a $\mathcal{D}$-valued 1-Čech-cocycle $(g_{ab} \in \mathcal{D})$, $a, b \in A$, on $\mathcal{V}$, satisfying a certain condition which expresses that the class of $\mathcal{D}$-isomorphic covering spaces to which it refers is simply connected (elements of Čech cohomology and its relation to covering spaces are explained in the appendix, sections C, D, E).

We now explain what this condition means. Let $p : X \to Y$ be a universal covering manifold of Y with base points $x \in X$ and $y \in Y$ such that $y = p(x)$. The deck transformation group $\mathcal{D}$ of such a covering is isomorphic to the fundamental group $\pi_1(Y, y)$, the isomorphism being defined as in the discussion preceding theorem 3.6: If $[\lambda]$ is any loop class in $\pi_1(Y, y)$, let $\tilde{\lambda}$ denote the unique lift of λ to x; then there exists a unique deck transformation γ such that $\gamma x = \tilde{\lambda}(1)$. This deck transformation is the image of $[\lambda]$ under the above-mentioned isomorphism. The manifold Y can be considered as the orbit space $X/\mathcal{D}$ with the quotient topology. We now identify $\mathcal{D}$ with the fundamental group $\pi_1(Y, y)$, so that Y can be regarded as the orbit space $X/\pi_1(Y, y)$, π_1 acting via deck transformations γ on X, and $p : X \to X/\mathcal{D} = Y$ is a $\mathcal{D}$-covering.

Next, we note that, although a simply connected covering manifold X of Y is uniquely defined up to $\mathcal{D}$-isomorphisms, in general there are also $\mathcal{D}$-coverings $q : Z \to Y = Z/\mathcal{D}$ of Y which are **not** connected. This means that Y can be expressed as an orbit space of different, not necessarily connected, manifolds Z (with base point z), under an action of $\mathcal{D}$. These manifolds can be assembled into equivalence classes, equivalence being expressed by $\mathcal{D}$-isomorphism (see appendix, section D), and a universal covering manifold X determines just one class in this collection. As explained in the appendix, section E, there is a bijection between these equivalence classes and the classes of cohomologous 1-Čech-cocycles on $\mathcal{V}$, in other words, the elements of $H^1(\mathcal{V}; \mathcal{D})$; hence the bijection

$$\{\mathcal{D}\text{-coverings with base point}\}/\text{isomorphism} \leftrightarrow H^1(\mathcal{V}; \mathcal{D}) \quad . \tag{12}$$

Furthermore, a result in the theory of covering spaces (see, e.g., [3]) states that there is a bijection between $\{\mathcal{D}\text{-coverings with base point}\}/\text{isomorphism}$ and the set $Hom(\mathcal{D}, \mathcal{D})$ of homomorphisms from $\mathcal{D} = \pi_1(Y, y) \to \mathcal{D}$. Hence we also have a bijection

$$H^1(\mathcal{V}; \mathcal{D}) \leftrightarrow Hom(\mathcal{D}, \mathcal{D}) \quad . \tag{13}$$

We explain (LHS→RHS) of this bijection. Let the Čech cocycle (g_{ab}) be given. We first show that a $\mathcal{D}$-covering Z of Y exists such that the Čech cocycle determined by a collection (i_a) of trivializations is the given one. To this end, consider the topological sum of all $\mathcal{D} \times V_a$ (i.e. the underlying set is a disjoint union) and define the relation $(d, y) \sim (d', y')$ for elements $(d, y) \in \mathcal{D} \times V_a$, $(d', y') \in \mathcal{D} \times V_b$ to be true if and only if $(d', y') = (d \cdot g_{ab}, y)$; then properties (Trans1-Trans3) in the appendix, section E, guarantee that " $\sim$ " is an equivalence relation. Now define Z to be the quotient $Z \equiv \sqcup_{a \in A} \mathcal{D} \times V_a / \sim$, endowed with the final topology (quotient topology). It is easy to see that Z is a $\mathcal{D}$-covering of Y, i.e. $Y = Z/\mathcal{D}$. The set of maps (i_a) that send elements $(d, y) \in \mathcal{D} \times V_a$ to the equivalence classes $i_a(d, y)$ to which they belong provides the natural collection of trivializations for this $\mathcal{D}$-space; it follows that $i_b^{-1} \circ i_a(d, y) = (d \cdot g_{ab}, y)$, so that the associated Čech cocycle is the one we have started with. Choose base points $z \in Z$, $y = q(z)$. Now observe that the group $\mathcal{D} \simeq \pi_1(Y, y)$ enters this construction in two different ways: Firstly, the elements of $\mathcal{D}$ locally label

the different sheets of the covering in a trivialization. Secondly, $\mathcal{D}$ is the set of homotopy classes of loops at y on the base manifold Y. We now construct a homomorphism ρ from $\mathcal{D}$ as the set of homotopic loops to $\mathcal{D}$ as the labelling space for the sheets of the covering: Choose a homotopy class $[\gamma]$, where γ is a loop at the base point $y \in Y$. The unit interval $[0,1]$ can be divided [3] into $0 = t_0 < t_1 < \cdots < t_n = 1$ such that the image of each interval $[t_{i-1}, t_i]$ lies in the open set $V_{a(i)}$. Then every point $\gamma(t_i)$ lies in $V_{a(i)} \cap V_{a(i+1)}$; on this domain, the cocycle $g_{a(i)a(i+1)}$ is constant. We lift the loop γ to a curve $\tilde{\gamma}$ starting at the base point z in Z. If $i_{a(0)}(d, y) = z = \tilde{\gamma}(0)$, then we find that

$$i_{a(0)}^{-1} \tilde{\gamma}(1) = \left(d \cdot g_{a(0)a(1)} \cdot \cdots \cdot g_{a(n)a(0)}, y\right) \equiv (d \cdot \rho[\gamma], y) \quad , \tag{14}$$

which defines an element $\rho[\gamma] \equiv g_{a(0)a(1)} \cdot \cdots \cdot g_{a(n)a(0)} \in \mathcal{D}$. It can be shown that this is independent of the representative γ of the homotopy class $[\gamma]$, and furthermore, that the assignment $[\gamma] \mapsto \rho[\gamma]$ is a homomorphism.

Thus, $\mathcal{D}$-coverings of Y, or equivalently, Čech cocycles on $\mathcal{V}$, are characterized by, and in turn characterize, homomorphisms $\mathcal{D} \to \mathcal{D}$. Cohomologous cocycles $g'_{ab} = h_a^{-1} g_{ab} h_b$ give rise to homomorphisms ρ, ρ' that differ by conjugation, i.e. an inner automorphism of $\mathcal{D}$, $\rho'[\gamma] = h_{a(0)}^{-1} \cdot \rho[\gamma] \cdot h_{a(0)}$. For example, the homomorphism $\mathcal{D} \to \mathcal{D}$ is the trivial one, i.e. $[\gamma] \mapsto e \in \mathcal{D}$ for all elements $[\gamma]$ in $\mathcal{D}$, if and only if the associated $\mathcal{D}$-covering of Y is (isomorphic to) the trivial $\#\mathcal{D}$-sheeted covering $\mathcal{D} \times Y \to Y$ of Y, $\mathcal{D}$ acting on $\mathcal{D} \times Y$ by left multiplication on the first factor. On the other hand, we now show that the class of $\mathcal{D}$-isomorphic simply connected $\mathcal{D}$-coverings $p : X \to Y = X/\mathcal{D}$ is characterized by homomorphisms ρ which are inner automorphisms of $\mathcal{D}$: To see this, we first examine the simply connected covering space X consisting of all homotopy classes $[\gamma]$ of curves γ in Y with initial point $\gamma(0) = y$, where y is the base point of Y. Choose trivializations i_a on $\mathcal{D} \times V_a$ so that the image $i_a^{-1}[c]$ of the constant loop c at y is represented by $([c], y)$ for all a for which $y \in V_a$. Then an arbitrary loop class $[\gamma] \in \pi_1(Y, y)$ is represented by $[\gamma] \equiv i_a([\gamma], y) \in X$; but this element is just the endpoint $\tilde{\gamma}(1)$ of the lift $\tilde{\gamma}$ of γ to $[c] \in p^{-1}(y)$, which implies by formula (14) that $\rho[\gamma] = [\gamma]$. Since this holds for all $[\gamma]$, we have $\rho = id|\,\mathcal{D}$ in this case. Now, if $p' : X' \to Y$ is another simply connected covering, we have seen above that the associated cocycles are cohomologous, hence the associated $\mathcal{D}$-homomorphisms differ by an inner automorphism; but since ρ is the identity, this means that every homomorphism $\rho' : \mathcal{D} \to \mathcal{D}$ must be an inner automorphism of $\mathcal{D}$.

The developments of the last paragraph together with the content of theorems 3.5 and 3.6 are summarized in

4.1 Theorem: Multi-valued potentials

Let α be a closed 1-form on the smooth manifold Y with base point y. Let $\mathcal{V} = \{V_a \mid a \in A\}$ be a simply connected path-connected open cover of Y. Let $\mathcal{D} \equiv \pi_1(Y, y)$. Then

(A) for every $\mathcal{D}$-valued 1-Čech-cocycle (g_{ab}), $a, b \in A$, on $\mathcal{V}$ whose associated homomorphism $\rho : \mathcal{D} \to \mathcal{D}$ is an *inner* automorphism of $\mathcal{D}$, i.e. $\rho(d') = d \cdot d' \cdot d^{-1}$ for some fixed $d \in \mathcal{D}$, there exists a collection of functions $f_{a,d} : V_a \to \mathbb{R}$ for $a \in A$, $d \in \mathcal{D}$, such that

1. $f_{a,d}$ is a local potential for α, i.e. $df_{a,d} = \alpha$ on V_a, for all $a \in A$ and $d \in \mathcal{D}$;
2. let λ be a loop at y with $[\lambda] = d \in \pi_1(Y, y) \simeq \mathcal{D}$. Then

$$f_{a,d} = f_{a,e} + \int\limits_{\lambda} \alpha \quad , \tag{15}$$

   where e is the identity in $\mathcal{D}$.
3. the $f_{a,d}$ satisfy a *glueing condition*, expressed by

$$f_{a,d} = f_{b,d \cdot g_{ab}} \tag{16}$$

   on $V_a \cap V_b \neq \emptyset$.

(B) Let (g'_{ab}) be a cocycle cohomologous to (g_{ab}), and let $\left(f'_{a,d}\right)$ be a collection of functions on $\mathcal{V}$ satisfying properties (A1–A3) with respect to (g'_{ab}). Then there exists a real constant c and a $\mathcal{D}$-valued 0-Čech cochain $(k_a : V_a \to \mathcal{D})$ on $\mathcal{V}$ such that

$$f'_{a,d} = f_{a,d \cdot k_a} + c \tag{17}$$

for all $a \in A$, $d \in \mathcal{D}$. The 0-cochain (k_a) is determined by the cocycles (g_{ab}) and (g'_{ab}) up to its value k_{a_0} on the open set $V_{a_0} \in \mathcal{V}$ which contains the base point y; on V_{a_0}, k_{a_0} can range arbitrarily in the coset $h_{a_0}^{-1} \cdot \mathcal{D}_{center}$, where h_{a_0} is the value of the Čech cochain which relates the cocycles (g_{ab}) and (g'_{ab}) on V_{a_0}, and $\mathcal{D}_{center}$ is the center of $\mathcal{D}$.

(C) **Definition:** A collection $(g_{ab}; f_{a,d})$ satisfying properties (A1–A3) will be called a *multi-valued potential function* for the closed 1-form α on Y.

Proof :

Ad (A) : Let X be the identification space $i_a : \mathcal{D} \times V_a \to X \equiv \bigsqcup_{a \in A} \mathcal{D} \times V_a / \sim$, where the relation $(d, y) \sim (d', y')$ for $(d, y) \in \mathcal{D} \times V_a$, $(d', y') \in \mathcal{D} \times V_b$ is defined to be true if and only if $(d', y') = (d \cdot g_{ab}, y)$, in which case these elements are identified according to $i_a(d, y) = i_b(d', y')$. Then X is a covering space of Y, and Y is the space of orbits on X under the action of $\mathcal{D}$ on X according to $(d', (d, y)) \mapsto (d' \cdot d, y)$. Since the homomorphism $\rho : \mathcal{D} \to \mathcal{D}$ associated with (g_{ab}) is an inner automorphims by assumption, it follows from the discussion at the beginning of this section that X is simply connected. Since Y is a smooth manifold, X is a smooth manifold. Let $p : X \to Y$ be the projection, which is a

local diffeomorphism. Then $p^*\alpha$ is a closed, hence exact, 1-form on X, and has a potential F with $dF = p^*\alpha$. The identification maps $i_a : \mathcal{D} \times V_a \to p^{-1}(V_a)$ are the natural trivializations for this covering. If we write $i_a(d,y) \equiv i_{a,d}(y)$, then $i_{a,d}$ is the inverse of the restriction $p|\, i_a(\{d\} \times V_a)$. Now define $f_{a,d}(y) \equiv F \circ i_a(d,y)$ for $y \in V_a$. By construction, $df_{a,d} = i^*_{a,d}dF$, and since $dF = [p|\, i_a(\{d\} \times V_a)]^* \alpha$, it follows that $df_{a,d} = \alpha$ on V_a. Furthermore, if also $y \in V_b$, then $F \circ i_a(d,y) = (F \circ i_b) \circ \left(i_b^{-1} \circ i_a\right)(d,y) = (F \circ i_b)(d \cdot g_{ab}, y) = f_{b,d\cdot g_{ab}}(y)$. Formula (15) is a consequence of theorems 3.5 and 3.6. This proves (A).

Ad (B) : From the cocycles (g_{ab}), (g'_{ab}), construct coverings $p : X \to Y$, $q : Z \to Y$ as in the proof of (A), with trivializations $i_a : \mathcal{D} \times V_a \to X$, $j_a : \mathcal{D} \times V_a \to Z$. Then both X and Z are smooth, simply connected manifolds. The glueing condition $f_{a,d} = f_{b,d\cdot g_{ab}}$ for $(f_{a,d})$ implies that there exists a smooth function F on X such that $F \circ i_{a,d} = f_{a,d}$: For, we have $f_{a,d}(y) = f_{b,d'}(y')$ whenever $i_a(d,y) = i_b(d',y')$; the universal property of the identification space [7] $i_a : \mathcal{D} \times V_a \to X$ guarantees the existence of a smooth F with the desired property. A similar function F' with $f' \circ j_{a,d} = f'_{a,d}$ exists on Z. Since X and Z are $\mathcal{D}$-isomorphic, there exists a diffeomorphism $\phi : Z \to X$ preserving fibres, i.e. $p \circ \phi = q$, and being $\mathcal{D}$-equivariant, i.e. $\phi(d \cdot z) = d \cdot \phi(z)$. In the trivializations employed above we have

$$i_a^{-1} \circ \phi \circ j_a(d,y) = (d \cdot k_a, y) \quad , \tag{18}$$

where the collection $(k_a : V_a \to \mathcal{D})$ defines a 0-Čech cochain on $\mathcal{V}$.

Since $d(F \circ \phi) = \phi^* dF = \phi^* p^* \alpha = q^* \alpha$, we see that both $F \circ \phi$ and F' are potentials for $q^*\alpha$ on Z; since Z is simply connected, it follows that $F' = F \circ \phi + c$ with $c \in \mathbb{R}$. Then

$$f'_{a,d}(y) = F' \circ j_a(d,y) = F \circ (\phi \circ j_a)(d,y) + c = (F \circ i_a)(d \cdot k_a, y) + c = f_{a,d\cdot k_a}(y) + c \quad ,$$

where we have used (18); thus (17) follows.

Furthermore, assume that $V_a \cap V_b \neq \emptyset$, then the analogue of (18) on V_b reads $i_b^{-1} \circ \phi \circ j_b(d \cdot g'_{ab}, y) = (d \cdot g'_{ab} \cdot k_b, y)$, which implies

$$\left(i_b^{-1} \circ i_a\right) \circ \left(i_a^{-1} \circ \phi \circ j_a\right) \circ \left(j_a^{-1} \circ j_b\right)(d \cdot g'_{ab}, y) = (d \cdot g'_{ab} \cdot k_b, y) \quad ,$$

from which it follows that

$$g'_{ab} \cdot k_b = k_a \cdot g_{ab} \quad . \tag{19}$$

This formula says that the cochain (k_a) is not arbitrary, but is completely determined by the cocycles (g_{ab}) and (g'_{ab}), once a choice has been made for the value of k_a on one selected V_a (e.g. the V_{a_0} which contains the base point y of Y). This follows from path-connectedness of Y; for, if V_{a_n} is any open set in $\mathcal{V}$, there exist finite sequences $g'_{a_0a_1}, \ldots, g'_{a_{n-1}a_n}$, and $g_{a_0a_1}, \ldots, g_{a_{n-1}a_n}$ so that

$$k_{a_n} = g'^{-1}_{a_{n-1}a_n} \cdots g'^{-1}_{a_0a_1} \cdot k_{a_0} \cdot g_{a_0a_1} \cdots g_{a_{n-1}a_n} \quad .$$

Since the selected k_{a_0} is constant on V_{a_0}, a choice of k_{a_0} is just a choice of an element of $\mathcal{D}$. Furthermore, if l is any loop in Y at the base point y representing

the element $\delta \in \mathcal{D}$, then from (14) we see that the associated series $\left(g_{a_i a_{i+1}}\right)$ and $\left(g'_{a_i a_{i+1}}\right)$ represent $\rho[l] \equiv g_{a(0)a(1)} \cdots g_{a(n)a(0)}$, $\rho'[l] \equiv g'_{a(0)a(1)} \cdots g'_{a(n)a(0)}$, respectively. Thus, we must have

$$k_{a_0} = \rho'[l]^{-1} \cdot k_{a_0} \cdot \rho[l]$$

for all homotopy classes $[l] \in \pi_1(Y, y) \equiv \mathcal{D}$. However, as $\mathcal{D}$ acts transitively on each fibre of the covering $p : X \to Y$, it follows that the elements $\rho'[l]$, $\rho[l]$ take any value in $\mathcal{D}$, as $[l]$ ranges in $\mathcal{D}$. Using $\rho'[l] = h_{a_0}^{-1} \cdot \rho[l] \cdot h_{a_0}$ it follows that $k_{a_0} = h_{a_0}^{-1} \cdot \rho[l]^{-1} \cdot h_{a_0} \cdot k_{a_0} \cdot \rho[l]$, or

$$\delta \cdot (h_{a_0} k_{a_0}) = (h_{a_0} k_{a_0}) \cdot \delta$$

for all $\delta \in \mathcal{D}$. But this implies that $h_{a_0} k_{a_0}$ must lie in the center of $\mathcal{D}$, which is a normal subgroup of $\mathcal{D}$. Therefore k_{a_0} can range in the coset $h_{a_0}^{-1} \cdot \mathcal{D}_{center}$. Hence the collection $(f_{a,d})$ is determined up to a real constant and an arbitrary element in $h_{a_0}^{-1} \cdot \mathcal{D}_{center}$. ∎

In the sequel we apply the covering techniques discussed so far to coverings of symplectic manifolds. We first present our notational conventions:

5 Notation and conventions for symplectic manifolds

Sections 5–6 are based on [2], [8], [10].

— We recall that a symplectic form ω on a manifold M is a closed, non-degenerate 2-form on M. In this case, the pair (M, ω) is called a symplectic manifold.

— By $\mathcal{F}(M)$ we denote the set of all smooth functions $f : M \to \mathbb{R}$. On a symplectic manifold we can make $\mathcal{F}(M)$ into a real Lie algebra using Poisson brackets.

— By $\chi(M)$ we denote the set of all smooth vector fields on M.

— If G is a Lie group, we will frequently denote its Lie algebra by $\hat{g}$, and the coalgebra, i.e. the space dual to $\hat{g}$, by g^*.

— Given an action $\phi : G \times M \to M$ of a Lie group G on a manifold M, we will frequently denote the components of its tangent map ϕ_* by $\phi_* = \left(\frac{\partial \phi}{\partial G}, \frac{\partial \phi}{\partial M}\right)$. If $A \in \hat{g}$, the induced vector field on M will be denoted by $\frac{\partial \phi}{\partial G} A$ or $\tilde{A}$.

— Interior multiplication of a vector V with a q-form ω will be denoted by $V \vdash \omega$.

— A diffeomorphism $f : M \to M$ on a symplectic manifold (M, ω) is called canonical transformation, if $f^* \omega = \omega$. An action $\phi : G \times M \to M$ of a Lie group G on M is called symplectic if every ϕ_g is a canonical transformation.

— I generally denotes the closed interval $I = [0, 1] \subset \mathbb{R}$.

6 Hamiltonian and locally Hamiltonian vector fields

On a symplectic manifold, the symplectic form ω provides a non-natural isomorphism between tangent spaces $T_x M$ and cotangent spaces $T_x^* M$ at every point $x \in M$, since ω is non-degenerate. In particular, for every $f \in \mathcal{F}(M)$ there exists a unique vector field $\rho f \in \chi(M)$ such that

$$\rho f \vdash \omega + df = 0 \quad . \tag{20}$$

This gives us a well-defined map $\rho : \mathcal{F}(M) \to \chi(M)$. A vector field $V \in \chi(M)$ which is the image of a function $f \in \mathcal{F}(M)$ under ρ, $V = \rho f$, is called Hamiltonian. The set of all (smooth) Hamiltonian vector fields on M is denoted by $\chi_H(M)$, and is a real vector space.

On the other hand, the set $\chi_{LH}(M)$ of vector fields V on M which satisfy

$$\mathcal{L}_V \omega = 0 \quad , \tag{21}$$

where $\mathcal{L}_V$ denotes a Lie derivative, is called the set of locally Hamiltonian vector fields. This means that on every simply connected open neighbourhood $U \subset M$, the 1-form $V \vdash \omega$ is exact, hence there exists a smooth function $f \in \mathcal{F}(U)$ such that

$$V \vdash \omega + df = 0 \quad \text{on } U \quad . \tag{22}$$

As (22) holds in a neighbourhood of every point, we refer to V as a locally Hamiltonian vector field. The functions $f \in \mathcal{F}(U)$ need not be globally defined. If M is simply connected, then every locally Hamiltonian vector field is Hamiltonian, and $\chi_{LH}(M) = \chi_H(M)$. This will not be true for the manifolds we are interested in in this work.

7 Cotangent bundles

In this section we compile some standard facts about cotangent bundles we shall use throughout this paper. This material is discussed in standard textbooks on Symplectic Geometry (e.g. [8]), Mechanics (e.g. [9]), Differential Geometry (e.g. [10]), and Algebraic Topology (e.g. [11]).

— On the cotangent bundle T^*M of a manifold M we have a projection $\tau : T^*M \to M$, and a natural symplectic 2-form being given as the differential $\omega = d\theta$ of the canonical 1-form [8], [10] θ on T^*M, which is defined as follows: For $V \in T_{(m,p)}T^*M$, the action of θ on V is defined by $\langle \theta, V \rangle (m,p) \equiv \langle p, \tau_* V \rangle$.

— The homotopy groups of T^*M are determined by those of M; in fact we have

$$\pi_n(T^*M) \simeq \pi_n(M) \tag{23}$$

for all $n \geq 0$. This follows from the exact homotopy sequence for fibrations (see any textbook on Algebraic Topology, e.g. [11]),

$$\cdots \to \pi_{n+1}(B) \xrightarrow{\partial} \pi_n(F) \xrightarrow{i_\#} \pi_n(E) \xrightarrow{p_\#} \pi_n(B) \to \cdots \quad , \tag{24}$$

where $E \xrightarrow{p} B$ is a fibration with standard fibre F. For a vector bundle with $F \simeq \mathbb{R}^k$, the homotopy groups $\pi_n\left(\mathbb{R}^k\right)$ are trivial, hence

$$0 \to \pi_n(E) \xrightarrow{p_\#} \pi_n(B) \to 0$$

is an exact sequence, which says that $p_\#$ is an isomorphism in this case. As a consequence, (23) follows.

A diffeomorphism $f : M \to M$ can be extended to a diffeomorphism ${}^*f : T^*M \to T^*M$ as follows [8]: For $(m, p) \in T^*M$, let

$$({}^*f)(m, p) \equiv \left(f(m), \left(f^{-1}\right)^* p\right) \quad . \tag{25}$$

${}^*f$ is fibre-preserving and hence a bundle map. Given two diffeomorphisms f, g we find

$${}^*(fg) = ({}^*f)({}^*g) \quad . \tag{26}$$

From definition (25) it follows immediately that every ${}^*f$ preserves the canonical 1-form θ,

$$({}^*f)^* \theta = \theta \quad . \tag{27}$$

8 Coverings of cotangent bundles

In this section we start with symplectic manifolds $(T^*Y, d\theta)$ which are the cotangent bundles T^*Y of non-simply connected configuration spaces Y, and are endowed with the natural symplectic 2-form $d\theta$ which is associated with the canonical 1-form θ on the cotangent bundle. Then we extend a simply connected covering manifold X of Y to a covering manifold T^*X of T^*Y and show how the projection map $p : X \to Y$ can be extended to give a local symplectomorphism between T^*X and T^*Y which is also a covering map.

8.1 Covering spaces and their cotangent bundles

Let $p : X \to Y$ be a covering of manifolds. Let Θ, θ denote the canonical 1-forms on the cotangent bundles T^*X, T^*Y, respectively (see section 7). The bundle projections are written as $\sigma : T^*X \to X$ and $\tau : T^*Y \to Y$; the canonical symplectic 2-forms on T^*X, T^*Y are $\Omega = d\Theta$, $\omega = d\theta$, respectively.

Central to our developments is the observation that we can extend the projection p to a covering map of cotangent bundles; for all coverings it is understood that they are smooth:

8.2 Theorem

Let $p : X \to Y$ be a covering. Then p can be extended to a bundle map ${}^*p : T^*X \to T^*Y$ such that ${}^*p : T^*X \to T^*Y$ is a covering.

Proof :

We first have to show that a well-defined extension ${}^*p$ exists. To this end, let $V \subset Y$ be an admissible open neighbourhood in Y, and let $U \subset X$ denote a connected component of $p^{-1}(V)$. On every U, the restriction $p|\,U : U \to Y$ is a diffeomorphism, hence its extension

$$^*\left(p|\,U\right) \equiv \left(p|\,U, [p|\,U]^{-1*}\right) \tag{28}$$

is well-defined. The collection of all U, being the collection of all inverse images of admissible open neighbourhoods V in Y forms an open cover of X. On the intersection of any two of the U, U', the locally defined maps (28) coincide. Now define ${}^*p$ to be the uniquely determined function on T^*X whose restriction to any of the U coincides with $^*\left(p|\,U\right)$.

From this it follows that ${}^*p$ is well-defined, and is smooth, provided p is smooth. Its local form (28) shows that ${}^*p$ preserves fibres and hence is a bundle map. If V is any admissible neighbourhood in Y, then $\bigcup\limits_{y \in V} T^*_y Y$ is a neighbourhood in T^*Y whose inverse image under ${}^*p$ is a disjoint union of neighbourhoods in T^*X. This says that ${}^*p$ is a covering map. ■

As a consequence of the last theorem, we can pull back q-forms ω on T^*Y to q-forms $({}^*p)^*\,\omega$ on T^*X via ${}^*p$. Hence the pull-back $({}^*p)^*\,\theta$ is well-defined. The next theorem explains the relation between this pull-back and the canonical symplectic potential Θ on T^*X. To this end we note that

8.3 Lemma

$$\tau \circ ({}^*p) = p \circ \sigma \quad . \tag{29}$$

Proof :

This follows from the definition $({}^*p)\,(x, \alpha) = \left(p\,(x)\,, (p|\,U)^{-1*}\,\alpha\right)$ for $(x, \alpha) \in T^*X$. ■

8.4 Theorem

The pull-back of the canonical 1-form θ on Y under ${}^*p$ coincides with the canonical 1-form Θ on X,

$$({}^*p)^*\,\theta = \Theta \quad . \tag{30}$$

Proof :

Let $(x,\alpha) \in T^*X$, let $V \in T_{(x,\alpha)}T^*X$. Then $({}^*p)(x,\alpha) = \left(p(x), (p|\,U)^{-1*}\alpha\right)$, where U is a neighbourhood of x in X on which the restriction of p is injective. Now if $({}^*p)_* V$ is paired with θ at the point $({}^*p)(x,\alpha) \in T^*Y$, we obtain $\langle \theta, ({}^*p)_* V\rangle = \left\langle (p|\,U)^{-1*}\alpha, \tau_* ({}^*p)_* V\right\rangle$; however, from (29) we have that $\tau_* ({}^*p)_* = p_* \sigma_*$, and hence

$$\left\langle (p|\,U)^{-1*}\alpha, \tau_* ({}^*p)_* V\right\rangle = \langle \alpha, \sigma_* V\rangle = \langle \Theta, V\rangle \quad ,$$

by the definition of Θ; this proves the theorem. ■

There is an immediate important consequence:

8.5 Corollary

The canonical symplectic 2-form $d\Theta$ on T^*X is the pull-back of the canonical symplectic 2-form $d\theta$ on T^*Y under ${}^*p$, and hence the map ${}^*p : T^*X \to T^*Y$ is a **local symplectomorphism**.

Proof :

This follows from $d\Theta = d\,({}^*p)\,\theta = ({}^*p)\,d\theta$. ■

Hence we can relate, and locally identify, the dynamics taking place on T^*Y to an associated dynamical system on the symplectic covering manifold T^*X; this is one of the major statements of this work. As was shown in (23), the homotopy groups of a cotangent bundle are isomorphic to those of its base space. In particular, if X is simply connected, the fundamental groups obey the relations

$$\pi_1(T^*X) = \pi_1(X) = 0 \quad , \quad \pi_1(T^*Y) = \pi_1(Y) = \mathcal{D} \quad , \tag{31}$$

and it follows that the deck transformation group $\mathcal{D}(T^*X)$ of the covering ${}^*p : T^*X \to T^*Y$ is just $\pi_1(T^*Y) = \mathcal{D}$. This will enable us to remove the multi-valuedness of a local moment map given on a cotangent space T^*Y by constructing the local symplectomorphism ${}^*p$, and then studying the associated dynamics on the simply connected symplectic covering space T^*X, on which every locally Hamiltonian vector field has a globally defined charge, and hence every symplectic group action has a global moment map; see section 14 for the details. Our next task therefore is to study how (Lie) group actions defined on Y (and, in turn, T^*Y) can be lifted to a covering space space X (and T^*X), in particular, when X is simply connected; this is done in sections 9 and 10 for general symplectic manifolds which are not necessarily cotangent bundles.

9 Lift of group actions under covering maps

In this section we prove a couple of theorems about the lifting of a left action ϕ of a Lie group G on a manifold Y to a covering manifold X. Here we examine existence and uniqueness of lifts; in the next section we examine under which conditions the group law of G is preserved under the lift.

Let $p : X \to Y$ be a covering, where X, Y are connected, let V be a manifold, let $f : V \to Y$. As explained in section 2, one calls a map $\tilde{f} : V \to X$ a *lift of f through p*, if $p \circ \tilde{f} = f$. For the sake of convenience, we establish a similar phrase for a related construction which will frequently appear in the following: If $f : Y \to Y$, we call $\hat{f} : X \to X$ a *lift of f to X* if

$$p \circ \hat{f} = f \circ p \quad . \tag{32}$$

We enhance this condition for the case that G is a group and $\phi : G \times Y \to Y$ is a left action of G on Y. In this case, we call a smooth map $\hat{\phi} : G \times X \to X$ a *lift of ϕ to X* if

(L1) $p \circ \hat{\phi} = \phi \circ (id_G \times p)$, and

(L2) $\hat{\phi}_e = id_X$,

where $\hat{\phi}_e$ denotes the map $x \mapsto \hat{\phi}_e(x) \equiv \hat{\phi}(e, x)$.

We remark that (L1) does **not** imply that p is G-equivariant (or a G-morphism). This is because G-equivariance requires X to be a G-space, i.e. a manifold with a smooth left **action of** G on it. This in turn means that the lift $\hat{\phi}$ must preserve the group law of G, i.e. $\widehat{\phi_{gh}} = \hat{\phi}_g \hat{\phi}_h$. We will see shortly that this is guaranteed only if G is connected. If G has several connected components, the lift can give rise to an extension $\tilde{G}$ of the original group by the deck transformation group $\mathcal{D}$; this is described in theorem 10.2. The matter of equivariance of the covering map p is taken up in section 11.

We note that if a map $\hat{\phi}$ satisfying (L1) exists, it is determined only up to a deck transformation γ; for if we define $\hat{\phi}' \equiv \hat{\phi}(id_G \times \gamma)$, $\hat{\phi}'$ also satisfies (L1); this is why we have to impose (L2) additionally. However, we show that if $\hat{\phi}$ exists, then it can always be assumed that it satisfies (L2):

9.1 Proposition

1. Let $\phi : G \times Y \to Y$ be a smooth left action of G on Y. If a smooth map $\hat{\phi} : G \times X \to X$ satisfying (L1) exists, it can always be redefined so that

$$\hat{\phi}_e = id_X \quad . \tag{33}$$

2. Every $\hat{\phi}_g$ is a diffeomorphism, and the assignment $(g, x) \mapsto \hat{\phi}_g^{-1}(x)$ is smooth.

Proof :

Ad (1) : By assumption, $p\hat{\phi}_e = \phi_e p = p$, hence $\hat{\phi}_e$ is a deck transformation γ of the covering. Now redefine $\hat{\phi} \mapsto \hat{\phi}' = \hat{\phi}\left(id_G \times \gamma^{-1}\right)$, then $\hat{\phi}'$ satisfies $p\hat{\phi}' = \phi\left(id_G \times p\right)$ and $\hat{\phi}'_e = id_X$.

Ad (2) : The map $\hat{\phi}_g \hat{\phi}_{g-1} : X \to X$ projects into p, hence is a deck transformation γ. Thus $\hat{\phi}_g^{-1} = \widehat{\phi_{g^{-1}}} \circ \gamma^{-1}$ is smooth, since γ^{-1} is smooth, hence $\hat{\phi}_g$ is a diffeomorphism. The assignment $(g,x) \mapsto \hat{\phi}_g^{-1}(x) = \widehat{\phi_{g^{-1}}} \circ \gamma^{-1}(x)$ is smooth, since the map $G \to G$, $g \mapsto g^{-1}$ is smooth. ∎

In the following we assume that X and Y are connected manifolds, and Y is not simply connected. We note that

9.2 Remark

The fundamental group of the Lie group G coincides with the fundamental group of its identity component,

$$\pi_n(G) = \pi_n(G_0) \quad . \tag{34}$$

Proof :

This follows from the exact homotopy sequence (24)

$$\cdots \to \pi_{n+1}(B) \xrightarrow{\partial} \pi_n(F) \xrightarrow{i_\#} \pi_n(E) \xrightarrow{p_\#} \pi_n(B) \to \cdots \quad ,$$

for the fibration $pr : G \to G/G_0 = Ds$. Put $B = Ds$, $F = G_0$, $E = G$, $\pi_n(Ds) = 0$ to obtain (34). ∎

Now we derive a necessary and sufficient condition for the existence of a lift satisfying (L1, L2) in terms of the fundamental group $\pi_1(G)$ of G. To prove this, we first need a

9.3 Lemma

Let X, Y be topological spaces, let $\lambda \times \mu$ be a loop in $X \times Y$ at (x,y), where λ is a loop in X at x, and μ is a loop in Y at y. Then $\lambda \times \mu$ is homotopic to a product of loops

$$\lambda \times \mu \sim (\{x\} \times \mu) * (\lambda \times \{y\}) \quad , \tag{35}$$

where " $\sim$ " means "homotopic", and " $*$ " denotes a product of paths.

Proof :

We explicitly give a homotopy effecting (35). Define $h : I \times I \to X \times Y$, $(s,t) \mapsto h(s,t) = h_s(t)$. Here s labels the loops $t \mapsto h_s(t)$, and t is the loop

parameter. Define

$$h(s,t) \equiv \begin{cases} \left(x, \mu\left(\frac{t}{1-\frac{s}{2}}\right)\right) & \text{for} \quad t \in \left[0, \frac{s}{2}\right) \quad . \\ \left(\lambda\left(\frac{t-\frac{s}{2}}{1-\frac{s}{2}}\right), \mu\left(\frac{t}{1-\frac{s}{2}}\right)\right) & \text{for} \quad t \in \left[\frac{s}{2}, 1-\frac{s}{2}\right) \quad . \\ \left(\lambda\left(\frac{t-\frac{s}{2}}{1-\frac{s}{2}}\right), y\right) & \text{for} \quad t \in \left[1-\frac{s}{2}, 1\right] \quad . \end{cases} \tag{36}$$

This is the required homotopy. ■

Now we can turn our theorem. To this end, consider a left action $\phi : G \times Y \to Y$, and choose $y \in Y$ fixed. By ϕ_y we denote the map $\phi_y : G \to Y$, $g \mapsto \phi_y(g) \equiv \phi(g, y)$. ϕ_y induces the homomorphism $\phi_{y\#} : \pi_1(G, e) \to \pi_1(Y, y)$. Furthermore, if $x \in p^{-1}(y)$ is any point in the fibre over y, then p induces a map $p_\# : \pi_1(X, x) \to \pi_1(Y, y)$. We now can state:

9.4 Theorem

Let $x \in X$ arbitrary, let $y = p(x)$. Let G be connected. Then the action $\phi : G \times Y \to Y$ possesses a unique lift $\hat{\phi} : G \times X \to X$ satisfying (L1, L2) if and only if

$$\phi_{y\#}\, \pi_1(G, e) \subset p_\#\, \pi_1(X, x) \quad . \tag{37}$$

Proof :

Assume a lift $\hat{\phi}$ exists. Then $\hat{\phi}$ is the unique lift of the map $G \times X \to Y$, $(g, x) \mapsto \phi(g, p(x))$ through p with the property $\hat{\phi}(e, x) = x$. According to the "Lifting Map Theorem" 2 it follows that

$$[\phi(id_G \times p)]_\#\, \pi_1(G \times X, (e, x)) \subset p_\# \pi_1(X, x) \quad . \tag{38}$$

Conversely, if (38) holds, then $\phi(id_G \times p)$ lifts uniquely to $\hat{\phi}$ satisfying (L1, L2). Hence we need only show the equivalence (37) $\Leftrightarrow$ (38).

$\underline{(37) \Leftarrow (38) :}$ Let λ be a loop in G at e. Then $\lambda \times \{x\}$ is a loop in $G \times X$ at (e, x), hence $t \mapsto \phi(\lambda(t), p(x)) = \phi_y \circ \lambda(t)$ is a loop in Y at y giving rise to the class

$$\phi_{y\#}[\lambda] = [\phi(id_G \times p)]_\# [\lambda \times \{x\}] \in$$
$$\in [\phi(id_G \times p)]_\#\, \pi_1(G \times X, (e, x)) \subset p_\# \pi_1(X, x) \quad ,$$

where the last inclusion follows from assumption, and hence (37) follows.

$\underline{(37) \Rightarrow (38) :}$ Let $\lambda \times \mu$ be a loop in $G \times X$ at (e, x). Then lemma 9.3 says that

$$\lambda \times \mu \sim (\{e\} \times \mu) * (\lambda \times \{x\}) \quad ,$$

hence

$$[\phi(id_G \times p)]\, \lambda \times \mu \sim [\phi(id_G \times p)](\{e\} \times \mu) * [\phi(id_G \times p)](\lambda \times \{x\}) =$$

$$= \phi(e, p\mu) * \phi(\lambda, p(x)) = (p\mu) * (\phi_y \lambda) \quad .$$

But by assumption (37), there exists a loop ρ in X at x such that $\phi_y \lambda \sim p\rho$. Then the last expression in the last line above becomes

$$(p\mu) * (\phi_y \lambda) \sim p(\mu * \rho) \quad ,$$

where $\mu * \rho$ is a loop in X at x. But this proves

$$[\phi(id_G \times p)] [\lambda \times \mu] \in p_{\#} \pi_1(X, x) \quad ,$$

where $[\lambda \times \mu]$ is the homotopy class of $\lambda \times \mu$, and hence (38) follows. ■

10 Preservation of the group law

In this section we show that a lift $\hat{\phi}$ to a simply connected covering space is not unique, if G is not connected. Every lift gives rise to an extension of the original group G by the group $\mathcal{D}$ of deck transformations of the covering. If $\mathcal{D}$ is Abelian, all these extensions are equivalent.

We start with theorem 9.4: This gives a condition for the existence of a smooth map $\hat{\phi}$ satisfying (L1,L2) under the assumption that G is connected. We now temporarily relax the last condition and allow G to be a Lie group with several connected components. In this case we simply assume that a lift $\hat{\phi}$ exists. As mentioned above, the lift $\hat{\phi}$ need not preserve the group law on G, in which case it is not an action of G on X. In particular, this means, that the set

$$\left\{ \hat{\phi}_g \mid g \in G \right\} \tag{39}$$

of diffeomorphisms $\hat{\phi}_g : X \to X$ is no longer a group. Clearly, we want to know under which circumstances the lift **is** an action of some group $\tilde{G}$. We examine this under the assumption that G is a semidirect product $G = G_0 \odot Ds$ of its identity component and a discrete factor Ds, where $G/G_0 \equiv Ds$, as G_0 is normal in G. In this case, every element of G has a unique expression (g, κ), where $g \in G_0$, $\kappa \in Ds$. The group law is

$$(g, \kappa)(g', \kappa') = (g \cdot a(\kappa) g', \kappa\kappa') \quad , \tag{40}$$

where $a(\kappa) : G_0 \to G_0$ is an outer automorphism of G_0, and $a : Ds \to Aut(G_0)$ is a representation of Ds in the automorphism group of G_0, as $a(\kappa\kappa') = a(\kappa) \circ a(\kappa')$, and $a(e) = id$. We first need a

10.1 Lemma

Let $\gamma \in \mathcal{D}$, $g, g' \in G$. Then the maps

$$\hat{\phi}_g \circ \gamma \circ \hat{\phi}_g^{-1} \quad , \quad \hat{\phi}_g \circ \hat{\phi}_{g'} \circ \widehat{\phi_{gg'}}^{-1} \tag{41}$$

are deck transformations.

Proof :

This follows immediately by applying p and using (L1,L2). ■

Thus, for a given $g \in G$, (41) defines a map

$$b\,(g) : \mathcal{D} \to \mathcal{D} \quad , \quad \gamma \mapsto b\,(g)\,\gamma \equiv \hat{\phi}_g \circ \gamma \circ \hat{\phi}_g^{-1} \quad , \tag{42}$$

which implies that $b\,(g) \in Aut\,(\mathcal{D})$ is a $\mathcal{D}$-automorphism. $b : G \to Aut\,(\mathcal{D})$ need not be a representation, however! The second expression in (41) defines a map

$$\Gamma : G \times G \to \mathcal{D} \quad , \quad (g, g') \mapsto \Gamma\,(g, g') \equiv \hat{\phi}_g \circ \hat{\phi}_{g'} \circ \widehat{\phi_{gg'}}^{-1} \quad . \tag{43}$$

These maps determine how the group law of G is changed under the lift $\hat{\phi}$; in particular (43) shows that Γ expresses the deviation of the lifted diffeomorphisms $\hat{\phi}_g$ from forming a group isomorphic to G. This is the content of the next

10.2 Theorem

Let $\phi : G \times Y \to Y$ be a smooth action of G on Y. Assume that a smooth lift $\hat{\phi}$ satisfying (L1,L2) exists. Then

1. the set $\left\{\hat{\phi}_g \mid g \in G\right\}$ is no longer a group in general; instead, the lift $\hat{\phi}$ provides an *extension $\tilde{G}$ of G by the deck transformation group $\mathcal{D}$*, i.e. $\mathcal{D}$ is normal in $\tilde{G}$, and $\tilde{G}/\mathcal{D} = G$. The extended group is given by the set

$$\tilde{G} = \left\{\gamma \circ \hat{\phi}_g \mid \gamma \in \mathcal{D}\,;\; g \in G\right\} \quad . \tag{44}$$

If the elements of this set are denoted as pairs, $\gamma \circ \hat{\phi}_g \leftrightarrow (\gamma, g)$, then the group law of $\tilde{G}$ is given by

$$(\gamma, g) \cdot (\gamma, g') = (\gamma \cdot b\,(g)\,\gamma' \cdot \Gamma\,(g, g')\,, gg') \quad , \tag{45}$$

and inverses are

$$(\gamma, g)^{-1} = \left(b\,(g)^{-1}\left[\gamma^{-1} \cdot \Gamma^{-1}\left(g, g^{-1}\right)\right], g^{-1}\right) = \tag{46}$$

$$= \left(\left[b\left(g^{-1}\right)\gamma \cdot \Gamma\left(g^{-1}, g\right)\right]^{-1}, g^{-1}\right) \quad . \tag{47}$$

The group law (45) expresses the non-closure of the set $\left\{\hat{\phi}_g \mid g \in G\right\}$ as discussed in (39), since now

$$(e, g) \cdot (e, g') = (\Gamma\,(g, g')\,, gg') \quad . \tag{48}$$

2. If $H, K \subset G$ are any connected subsets of G, then the restrictions $b|\, H$ and $\Gamma|\, H \times K$ are constant. Hence both b and Γ descend to the quotient $G/G_0 = Ds$,

$$b : Ds \to \mathcal{D} \quad , \quad \Gamma : Ds \times Ds \to \mathcal{D} \quad . \tag{49}$$

3. In particular, on the identity component G_0 we have $b|\, G_0 = id$ and $\Gamma|\, G_0 \times G_0 = e$. Hence, for elements $(\gamma, g) \in \mathcal{D} \times G_0$ we have the group law

$$(\gamma, g) \cdot (\gamma', g') = (\gamma\gamma', gg') \quad . \tag{50}$$

As a consequence, the identity component G_0 can be regarded as a subgroup of the extension $\tilde{G}$, and can be identified with the set of all elements of the form (e, g), $g \in G_0$, so that (cf. 48)

$$(e, g) \cdot (e, g') = (e, gg') \quad . \tag{51}$$

Also, (50) says that $\mathcal{D}$ and G_0 commute. As a consequence of all that, the identity component $\tilde{G}_0$ of $\tilde{G}$ coincides (up to isomorphism) with the identity component of G, $\tilde{G}_0 = G_0$, and hence we have an isomorphism of Lie algebras

$$Lie\left(\tilde{G}\right) = Lie\,(G) = \hat{g} \quad . \tag{52}$$

Proof :

Ad (1) : We first must verify that the set $\tilde{G}$ as defined in (44) is indeed a group of diffeomorphisms on X. Since $\hat{\phi}_e = id_X$ by (L2), we have that $\tilde{G}$ contains $e\hat{\phi}_e = id_X$. Next, if $\gamma, \gamma' \in \mathcal{D}$ and $g, g' \in G$, then $\gamma\hat{\phi}_g\gamma'\hat{\phi}_{g'} = \gamma \circ b\,(g)\,\gamma' \circ \hat{\phi}_g \circ \hat{\phi}_{g'}$ according to (42); but $\hat{\phi}_g \circ \hat{\phi}_{g'} = \Gamma\,(g, g') \circ \widehat{\phi_{gg'}}$ by (43), which gives

$$\gamma \circ \hat{\phi}_g \circ \gamma' \circ \hat{\phi}_{g'} = [\gamma \circ b\,(g)\,\gamma' \circ \Gamma\,(g, g')] \circ \left[\widehat{\phi_{gg'}}\right] \quad , \tag{53}$$

where the first factor in square brackets on the RHS is an element of $\mathcal{D}$, and the second factor is a lifted diffeomorphism, and therefore the LHS is an element of the set $\tilde{G}$. Furthermore, when using the pair notation (γ, g) for elements of $\tilde{G}$, formula (53) yields the group law (45). Finally, we must show that inverses exist in $\tilde{G}$; it is easy to use (45) to arrive at (46) for inverses, which means that inverses have the form $\gamma \circ \hat{\phi}_g$ as required. Furthermore, from (45) it follows that $\mathcal{D}$ is normal in $\tilde{G}$, and that the cosets $\mathcal{D} \cdot (\gamma, g)$ obey the group law of G, since

$$[\mathcal{D} \cdot (e, g)]\,[\mathcal{D} \cdot (e, g')] = \mathcal{D} \cdot [(e, g)\,(e, g')] = \mathcal{D} \cdot (\Gamma\,(g, g')\,, gg') = \mathcal{D} \cdot (e, gg') \quad ,$$

thus $\tilde{G}/\mathcal{D} = G$.

Ad (2+3) : Let $\gamma \in \mathcal{D}$ be arbitrary, and consider the map $G \to p^{-1}(p(x))$, $g \mapsto [b(g)\gamma]\, x$, where the fibre $p^{-1}(p(x))$ is a discrete space for arbitrary fixed $x \in X$. From the definition (42) and proposition 9.1 we see that this map is smooth; since the target space is discrete, it must therefore be constant on every connected subset of the domain. Since X is connected, every deck transformation of the covering $p : X \to Y$ is uniquely determined by its value at a single point $x \in X$. Hence $b(g)\gamma = const.$ for all g within a connected component of G. On the identity component G_0, $b(g) = b(e) = id_{\mathcal{D}}$. – A similar argument applies to Γ, since (43) shows that all maps involved in the definition of Γ are smooth in all arguments. In particular, $\Gamma(e,g) = \Gamma(g,e) = e_{\mathcal{D}}$. (50) is a consequence of (45).

■

If $\mathcal{D}$ is Abelian, the map Γ defined in (43) has a special significance:

10.3 Theorem

If $\mathcal{D}$ is Abelian, then $\Gamma : G \times G \to \mathcal{D}$ defines a 2-cocycle in the $\mathcal{D}$-valued cohomology on G as defined in appendix, section F.

Proof :

The proof is a standard argument: Compute $\left(\hat{\phi}_g \hat{\phi}_h\right)\hat{\phi}_k = \hat{\phi}_g\left(\hat{\phi}_h \hat{\phi}_k\right)$ using (43); this leads to

$$b(g)\,\Gamma(h,k) + \Gamma(g,hk) - \Gamma(gh,k) - \Gamma(g,h) = (\delta\Gamma)(g,h,k) = 0 \quad ,$$

where we have used (112). ■

— Any two lifts $\hat{\phi}$ and $\hat{\phi}'$ must coincide on the identity component G_0, by uniqueness. However, they may differ on the components $G_0 \cdot \kappa$, $\kappa \in Ds$, by deck transformations. This gives rise to different cocycles (in case that $\mathcal{D}$ is Abelian) Γ and Γ'. We now show that Γ and Γ' must differ by a coboundary:

10.4 Theorem

Let the deck transformation group $\mathcal{D}$ of the covering $p : X \to Y$ be Abelian. Given two lifts $\hat{\phi}$ and $\hat{\phi}'$ of the action $\phi : G \times Y \to Y$ to X, with associated cocycles Γ and Γ', there exists a 1-cochain $\eta \in C^1(G,\mathcal{D})$ in the $\mathcal{D}$-valued cohomology on G as defined in Appendix, section F, such that

$$\Gamma' = \Gamma + \delta\eta \quad . \tag{54}$$

On the connected component G_0, $\eta = e$.

Proof :

Both $\hat{\phi}$ and $\hat{\phi}'$ project down to ϕ by (L1), which implies that $\hat{\phi}'_g \hat{\phi}_g^{-1}$ is a deck transformation $\eta(g)$, hence

$$\hat{\phi}'_g = \eta(g)\,\hat{\phi}_g \quad . \tag{55}$$

η is constant on connected components, hence is trivial on G_0. We have $\hat{\phi}'_g \hat{\phi}'_h = \Gamma'(g,h)\,\widehat{\phi'_{gh}}$, and inserting (55) gives

$$\eta(g) \cdot [b(g)\,\eta(h)] \cdot \Gamma(g,h) \cdot \widetilde{\phi_{gh}} = \Gamma'(g,h) \cdot \eta(gh) \cdot \widetilde{\phi_{gh}} \quad ;$$

hence, on using additive notation for the group composition in $\mathcal{D}$,

$$\Gamma'(g,h) = \Gamma(g,h) + \underbrace{b(g)\,\eta(h) - \eta(gh) + \eta(g)}_{=(\delta\eta)(g,h)} \quad , \tag{56}$$

using (112). ■

Thus, all the different lifts $\hat{\phi}$ of the group action ϕ to X give rise to the same cohomology class $[\Gamma] \in H^2(G,\mathcal{D})$ in the $\mathcal{D}$-valued cohomology on G, which also implies, that the possible group extensions $\tilde{G}$ of G associated with these lifts are all equivalent [12]. Thus, up to equivalence, there is only one group extension $\tilde{G}$ of G by $\mathcal{D}$, and this is basically determined by the geometry of the covering $p : X \to Y$.

11 $\tilde{G}$-spaces and equivariant covering maps

Although property (L1), $p\hat{\phi}_g = \phi_g p$, seems to suggest that every covering map p is a G-morphism, this is not true in general, if G is not connected; for in this case, X is not a G-space, but only a $\tilde{G}$-space, as explained above. However, we can make p equivariant with respect to the larger group $\tilde{G}$: To this end, we first note that this requires X and Y to be $\tilde{G}$-spaces. This can be accomplished by introducing the projection $pr : \tilde{G} \to \tilde{G}/\mathcal{D} = G$ and observing that Y is trivially a $\tilde{G}$-space by defining the action of $\tilde{G}$ on Y as

$$\Phi : \tilde{G} \times Y \to Y \quad , \quad \Phi \equiv \phi\,(pr \times id_Y) \quad . \tag{57}$$

Now we must define a suitable action $\hat{\Phi}$ of $\tilde{G}$ on X; this is accomplished by the assignment

$$\hat{\Phi} : \tilde{G} \times X \to X \quad , \quad ((\gamma,g),x) \mapsto \hat{\Phi}((\gamma,g),x) \equiv \gamma \circ \hat{\phi}_g(x) \quad . \tag{58}$$

Using the group law (45) for the case that $\mathcal{D}$ is Abelian we see that

$$\hat{\Phi}_{(\gamma,g)}\hat{\Phi}_{(\gamma',g')} = \hat{\Phi}_{(\gamma+b(g)\gamma'+\Gamma(g,g'),gg')} = \hat{\Phi}_{(\gamma,g)(\gamma',g')} \quad , \tag{59}$$

i.e. $\hat{\Phi}$ is indeed a left action. Furthermore, under the projection p we have

$$p\hat{\Phi}((\gamma,g),x) = p\hat{\phi}_g(x) = \phi(g,p(x)) = \phi(pr \times p)((\gamma,g),x) \quad ,$$

or

$$p \circ \hat{\Phi} = \phi \circ (pr \times p) = \Phi \circ (id_{\tilde{G}} \times p) \quad . \tag{60}$$

But this equation now says that the covering projection p is a $\tilde{G}$-morphism with respect to the group $\tilde{G}$, where X and Y are now regarded as $\tilde{G}$-spaces. Let us summarize:

11.1 Theorem: Equivariance of the covering map

Define actions $\Phi, \hat{\Phi}$ of the extended group $\tilde{G}$ on Y, X according to formulas (57), (58). With this definition, X and Y become $\tilde{G}$-spaces, and the covering map $p : X \to Y$ is $\tilde{G}$-equivariant (a $\tilde{G}$-morphism).

— We now discuss a situation in which a lift $\hat{\phi}$ as introduced in section 9 always exists. A glance at formula (37) in section 9.4 shows that the lift exists if the fundamental groups of both G and X are trivial. Now we perform the following construction: We assume that G is connected and X is simply connected. If G acts on Y via ϕ, then so does the universal simply connected covering group CG of G; just let $pro : CG \to G$ be the natural projection (which is a homomorphism), then $c\phi : CG \times Y \to Y$, $c\phi \equiv \phi\,(pro \times id_Y)$ defines a smooth left action of CG on Y. Therefore, in this case there always exists a lift $\widehat{c\phi} : CG \times X \to X$ of $c\phi$ to X satisfying (L1,L2). Since G is connected, formula (51) in theorem 10.2 tells us that the lift preserves the group law of CG, and hence $\widehat{c\phi}$ is an action of CG on X. Thus,

11.2 Theorem

Let $p : X \to Y$ be a covering, where X is simply connected. Let the connected Lie group G act on Y via ϕ, and let CG denote the universal covering group of G with projection homomorphism $pro : CG \to G$. Then $c\phi \equiv \phi\,(pro \times id_Y)$ defines an action of CG on Y, and there exists a lift $\widehat{c\phi} : CG \times X \to X$ such that $\widehat{c\phi}$ preserves the group law on CG,

$$\widehat{c\phi_{gh}} = \widehat{c\phi_g} \circ \widehat{c\phi_h} \quad . \tag{61}$$

Thus, the projection map p is a CG-morphism of CG-spaces,

$$p \circ \widehat{c\phi_g} = c\phi_g \circ p \quad , \tag{62}$$

for $g \in CG$.

— A consequence of the developments in this section is this: Assume that $p : X \to Y$ is a smooth covering of manifolds, where Y is a symplectic manifold with symplectic 2-form ω. Since p is a local diffeomorphism, the 2-form $\Omega \equiv p^*\omega$ is closed and non-degenerate, and hence is a valid symplectic 2-form on X. Now assume that a connected Lie group G acts on Y via $\Phi : G \times Y \to Y$ such that all diffeomorphisms Φ_g are canonical transformations, $\Phi^*\omega = \omega$. Assume that

a lift $\tilde{\Phi}$ to X exists; as G is connected, the group law is then preserved. Since the lift obeys $p \circ \tilde{\Phi}_g = \Phi_g \circ p$, it follows that $\tilde{\Phi}_g^* p^* \omega = p^* \Phi_g^* \omega$, or $\tilde{\Phi}_g^* \Omega = \Omega$; hence all diffeomorphisms $\tilde{\Phi}_g$ are canonical transformations on the symplectic covering manifold X. In summary,

11.3 Theorem

Let $p : X \to Y$ be a covering of smooth manifolds, where Y is a symplectic manifold with symplectic 2-form ω. Assume the connected Lie group G acts via Φ on Y from the left, such that all Φ_g are canonical transformations with respect to ω. Assume a lift $\tilde{\Phi}$ to X exists. Then all diffeomorphisms $\tilde{\Phi}_g$ are canonical transformations with respect to the symplectic 2-form $\Omega \equiv p^* \omega$ on X.

— Finally, we wish to study the following problem: Given a non-simply connected manifold Y, thought of as the "configuration space" of a dynamical system (or rather, a class of dynamical systems), and the left action $\phi : G \times Y \to Y$ of a (Lie) group G on Y; if a lift $\hat{\phi}$ exists, we can extend it to a map ${}^*\hat{\phi} : G \times T^*X \to T^*X$ on the cotangent bundle of X using formula (25). On the other hand, from the discussion in section 8.2 we know that T^*X is itself a covering space of T^*Y, and ${}^*p$ as defined in equation (28) is the covering map. Hence we can extend the action ϕ first to an action ${}^*\phi$ on the cotangent bundle T^*Y, and then ask whether a lift $\widehat{{}^*\phi}$ of ${}^*\phi$ to T^*X exists, and if yes, whether it coincides with the extension ${}^*\hat{\phi}$ of the lift $\hat{\phi}$ of ϕ. The next theorem gives the answer for connected G:

11.4 Theorem

Let G be a connected Lie group; let $p : X \to Y$ be a covering, where X is connected. Let $\phi : G \times Y \to Y$ be a smooth action of G on Y, which possesses a lift $\hat{\phi} : G \times X \to X$ satisfying (L1, L2). Then (1) a lift $\widehat{{}^*\phi} : G \times T^*X \to T^*X$ of the extension ${}^*\phi : G \times T^*Y \to T^*Y$ of the action ϕ to the cotangent bundle T^*X of X exists; and (2) this lift coincides with the uniquely determined extension ${}^*\widehat{\phi} : G \times T^*X \to T^*X$ of the lift $\hat{\phi} : G \times X \to X$ of ϕ to the cotangent bundle T^*X of X; i.e.

$$\widehat{{}^*\phi} = {}^*\widehat{\phi} \quad , \tag{63}$$

yielding a commutative diagram.

Proof :

Let $\hat{\phi}$ be the lift of ϕ satisfying (L1,L2). Its extension to the cotangent bundle T^*X takes the form ${}^*\widehat{\phi} = \left(\hat{\phi}, \hat{\phi}^{-1*}\right)$. This map satisfies

$${}^*\widehat{\phi_e} = id_{T^*X} \quad , \tag{64}$$

and under the extended projection ${}^*p$ it behaves as

$${}^*p \circ {}^*\hat{\phi} = {}^*\phi \circ (id_G \times {}^*p) \quad . \tag{65}$$

But formulas (64,65) are precisely the conditions (L1, L2) for a lift $\widehat{{}^*\phi}$ of the extended action ${}^*\phi$ under the covering ${}^*p : T^*X \to T^*Y$. This means that (1) ${}^*\hat{\phi}$ is a lift of ${}^*\phi$; and, since G and X, and hence $G \times T^*X$, are connected, this lift is unique as a consequence of the "Unique Lifting Theorem" 2, and hence coincides with $\widehat{{}^*\phi}$. ■

12 Symplectic G-actions and moment maps

In the following sections we generalize the usual definition of a moment map to a construction we call local moment map. The standard definitions found in the literature generally involve that a moment map is a **globally** defined function from a symplectic manifold M into the coalgebra of the Lie algebra of a Lie group G which acts on M symplectically (e.g. [9]). Other authors (e.g. [8], [2]) give even more restrictions by introducing moment maps only together with the condition that the first and second Chevalley-Eilenberg cohomology groups of the Lie group G are trivial, or equivalently, that the associated Lie algebra cohomoloy groups $H_0^1(\hat{g}, \mathbb{R})$ and $H_0^2(\hat{g}, \mathbb{R})$ are trivial. In this work we make no assumptions about cohomologies on the group G, nor do we assume that moment maps exist globally; on the contrary, it is the purpose of this work to generalize moment maps to situations where the underlying symplectic manifold is non-simply connected, and hence in general does not admit a global moment map.

Let M be a symplectic manifold with symplectic form ω. Let G be a Lie group, let $\Phi : G \times M \to M$ be a smooth symplectic left action of G on M, i.e. $\Phi(g, x) = \Phi_g(x)$, with $\Phi_{gg'} = \Phi_g \Phi_{g'}$, $\Phi_e = id_M$, and $\Phi_g^* \omega = \omega$ for all g. Let $\hat{g}$ be the Lie algebra of G, and g^* denote the coalgebra. If $A \in \hat{g}$, then the vector field induced by A on M is denoted $\frac{\partial \Phi}{\partial G} A \equiv \tilde{A}$. Since Φ preserves the symplectic form, the Lie derivative of ω with respect to $\tilde{A}$ vanishes, hence $\tilde{A}$ is a locally Hamiltonian vector field according to (21). From this it follows that the 1-form $\frac{\partial \Phi}{\partial G} A \vdash \omega$ is closed.

13 Global moment maps

First we assume that M is simply connected. By a standard argument, an $\mathbb{R}$-linear map f from a vector space $\hat{g}$ to the space of smooth closed 1-forms $Z^1_{deRham}(M)$ on M can always be lifted to an $\mathbb{R}$-linear map $h : \hat{g} \to \mathcal{F}(M)$, $A \mapsto h_A$ such that $dh_A = f(A)$: For, since M is simply connected, every closed 1-form $f(A)$ has a potential h_A with $dh_A = f(A)$; the assignment $(A, x) \mapsto h_A(x)$ can be assumed to be smooth in x, but need not be smooth in A. Now choose an arbitrary fixed point $x_0 \in M$, and replace h_A by $h_A - h_A(x_0)$; then $A \mapsto h_A(x) - h_A(x_0)$ is linear in A.

In particular, the map $\hat{g} \ni A \mapsto \frac{\partial \Phi}{\partial G} A \vdash \omega \in Z^1(M)$ can be lifted to an

$\mathbb{R}$-linear map $h : \hat{g} \to \mathcal{F}(M)$, $A \mapsto -h_A$, with

$$\frac{\partial \Phi}{\partial G} A \vdash \omega + dh_A = 0 \quad . \tag{66}$$

Since $A \mapsto h_A(x)$ is linear for every fixed $x \in M$, h defines a map $J : M \to g^*$, $\langle J(x), A \rangle \equiv h_A(x)$. J is called a *moment map* associated with the action Φ. From its definition via h we see that J is determined up to addition $J \mapsto J + L$ of an M-constant, $\mathbb{R}$-linear map $L : \hat{g} \to \mathbb{R}$, with $dL = 0$.

h as defined above is a homomorphisms of vector spaces by linearity; in general it is not a homomorphism of Lie algebras, however. Rather, the algebra of Poisson brackets can provide a central extension of the Lie algebra of the Hamiltonian vector fields [8].

14 Local moment maps

Now we assume that Y is connected, but not simply connected. If y is a base point of Y, set $\mathcal{D} \equiv \pi_1(Y, y)$. Let $\mathcal{V} = \{V_a \mid a \in A\}$ be a countable simply connected open cover of Y as introduced in section 4. For every $A \in \hat{g}$, $\frac{\partial \Phi}{\partial G} A \vdash \omega$ is a closed 1-form on Y; hence a multi-valued potential function $(h_{A,a,d})$, $d \in \mathcal{D}$, exists for every Čech cocycle (g_{ab}) associated with a simply connected cover of Y, according to theorem 4.1. However, here we can no longer be sure whether all $h_{A,a,d}$ can be made linear in A simultaneously, without spoiling the glueing conditions $h_{A,a,d} = h_{A,b,d \cdot g_{ab}}$. We therefore have to formulate the problem in terms of an appropriate covering space X, which we take, as in the proof of theorem 4.1, to be the identification space $i_a : \mathcal{D} \times V_a \to X \equiv \bigsqcup_{a \in A} \mathcal{D} \times V_a / \sim$, where $\sim$ relates elements (d, y) and $(d', y') = (d \cdot g_{ab}, y)$ on $\mathcal{D} \times V_a$ and $\mathcal{D} \times V_b$ that are identified as $i_a(d, y) = i_b(d', y')$. Then X is a universal covering space of Y, such that the projection $p : X \to Y$ is a local diffeomorphism. Y is the space of orbits under the action of $\mathcal{D}$ on X. If ω denotes the symplectic 2-form on Y, $\Omega \equiv p^* \omega$ is a symplectic 2-form on X.

We must ask whether Φ can be lifted to X, i.e. whether there exists a map $\hat{\Phi} : G \times X \to X$ satisfying $p \circ \hat{\Phi} = \hat{\Phi} \circ (id_G \times p)$, and $\hat{\Phi}_e = id_X$. Furthermore, one has to examine whether the group law is preserved by the lift, i.e. whether $\widehat{\Phi_{gh}} = \hat{\Phi}_g \hat{\Phi}_h$. The question of existence of a lift is examined in theorem 9.4; in theorem 10.2 it is proven that, for connected G, the lift preserves the group law of G. In theorem 11.3 it is proven that, for connected G, the diffeomorphisms $\hat{\Phi}_g$ are canonical transformations with respect to Ω, hence $\hat{\Phi}$ is a symplectic action. In this case we have a left action $\hat{\Phi}$ of G on the simply connected manifold X, so that the results from section 13 can be applied: There exists a lift of the $\mathbb{R}$-linear map $\hat{g} \ni A \mapsto \frac{\partial \hat{\Phi}}{\partial G} A \vdash \Omega \in Z^1_{deRham}(X)$ to an $\mathbb{R}$-linear map $\hat{h} : \hat{g} \to \mathcal{F}(X)$, $A \mapsto -\hat{h}_A$, with

$$\frac{\partial \hat{\Phi}}{\partial G} A \vdash \Omega + d\hat{h}_A = 0 \quad , \tag{67}$$

and there exists a global moment map $\hat{J}: X \to g^*$, $\left\langle \hat{J}(x), A \right\rangle \equiv \hat{h}_A(x)$. $\hat{J}$ is determined up to addition $\hat{J} \mapsto \hat{J} + L$ of an X-constant, $\mathbb{R}$-linear map $L : \hat{g} \to \mathbb{R}$, with $dL = 0$. Now we can set $h_{a,d}(y) \equiv \hat{h} \circ i_a(d, y)$, and $J_{a,d}(y) \equiv \hat{J} \circ i_a(d, y)$ for $y \in V_a$, $d \in \mathcal{D}$. Then

$$\frac{\partial \Phi}{\partial G} A \vdash \omega + d \langle J_{a,d}, A \rangle = 0 \tag{68}$$

on V_a, and for all $d \in \mathcal{D}$. The glueing condition is easily derived as

$$J_{a,d}(y) = \hat{J} \circ i_a(d, y) = \left(\hat{J} \circ i_b \right) \circ \left(i_b^{-1} \circ i_a \right)(d, y) =$$

$$= \left(\hat{J} \circ i_b \right)(d \cdot g_{ab}, y) = J_{b, d \cdot g_{ab}}(y) \quad . \tag{69}$$

— Now we examine to which extent a collection $(J_{a,d})$ is determined by the action Φ and a cocycle: Let (g_{ab}), (g'_{ab}) be cocycles such that the associated homomorphisms ρ, ρ' are inner automorphisms of $\mathcal{D}$, hence give rise to simply connected coverings; and let $(J_{a,d})$, $\left(J'_{a,d}\right)$ be collections satisfying relations (68,69), respectively. Then (g_{ab}) and (g'_{ab}) are cohomologous. As above, construct smooth simply connected covering manifolds $p : X \to Y$, $q : Z \to Y$ as identification spaces; the trivializations (i_a) with respect to X identify $i_a(d, y) = i_b(d \cdot g_{ab}, y)$, and the trivializations (j_a) with respect to Z identify $j_a(d, y) = j_b(d \cdot g'_{ab}, y)$. A lift $\tilde{\Phi}$ of Φ to Z exists precisely when a lift $\hat{\Phi}$ of Φ to X exists. The glueing conditions (69) for $(J_{a,d})$, $\left(J'_{a,d}\right)$ guarantee that there exist smooth functions $\hat{h}_A$, $\tilde{h}_A$ on X, Z, obeying

$$\hat{h}_A \circ i_{a,d} = \langle J_{a,d}, A \rangle \quad \text{and} \quad \tilde{h}_A \circ j_{a,d} = \left\langle J'_{a,d}, A \right\rangle \quad . \tag{70}$$

These functions define global moment maps $\left\langle \tilde{J}, A \right\rangle = \tilde{h}_A$ on Z and $\left\langle \hat{J}, A \right\rangle = \hat{h}_A$ on X. $\tilde{J}$ satisfies the analogue of (67),

$$\frac{\partial \tilde{\Phi}}{\partial G} A \vdash \Omega' + d\tilde{h}_A = 0 \tag{71}$$

on Z, where $\Omega' \equiv q^* \omega$. Furthermore, Z and X are $\mathcal{D}$-isomorphic, the isomorphism being effected by a diffeomorphism $\phi : Z \to X$, where ϕ preserves fibres, $p \circ \phi = q$, and is $\mathcal{D}$-equivariant. We examine the relation between the lifts $\hat{\Phi}$ and $\tilde{\Phi}$: Define a map $\psi : G \times Z \to Z$,

$$(g, z) \mapsto \psi(g, z) \equiv \phi^{-1} \circ \hat{\Phi}(g, \phi(z)) \quad . \tag{72}$$

A calculation shows that ψ satisfies $q \circ \psi = \Phi \circ (id_G \times q)$, and $\psi_e = id_Z$. But, as discussed in section 10, there exists precisely one function $G \times Z \to Z$ with

these two properties, and this is just the lift $\tilde{\Phi}$! Hence we deduce that $\psi = \tilde{\Phi}$, or

$$\phi \circ \tilde{\Phi} = \hat{\Phi} \circ (id_G \times \phi) \quad , \tag{73}$$

which gives a commutative diagram. It follows that

$$\frac{\partial \hat{\Phi}}{\partial G} = \phi_* \frac{\partial \tilde{\Phi}}{\partial G} \quad . \tag{74}$$

The symplectic 2-forms on X and Z are $\Omega \equiv p^*\omega$ and $\Omega' \equiv q^*\omega$, where ω is the symplectic 2-form on Y. From $p\phi = q$ we infer that $\Omega' = \phi^*\Omega$, hence ϕ is a symplectomorphism. From theorem 11.3 we know that $\hat{\Phi}$, $\tilde{\Phi}$ are symplectic actions with respect to Ω, Ω'.

If we insert (74) into (67) we obtain

$$\left[\phi_* \frac{\partial \tilde{\Phi}}{\partial G} A\right] \vdash \Omega + d\hat{h}_A = 0 \quad ; \tag{75}$$

using the relation $\Omega' = \phi^*\Omega$ and (71) we deduce $\left[\phi_* \frac{\partial \tilde{\Phi}}{\partial G} A\right] \vdash \Omega = \phi^{-1*}\left[\frac{\partial \tilde{\Phi}}{\partial G} A \vdash \Omega'\right]$, which, together with (75), yields $d\left\langle \tilde{J}, A\right\rangle = \phi^* d\left\langle \hat{J}, A\right\rangle$, or

$$\tilde{J} = \hat{J} \circ \phi + L \quad , \tag{76}$$

where L is a Z-constant linear map $\hat{g} \to \mathbb{R}$. Using the trivializations (i_a), (j_a), (70) and (76) we have

$$J'_{a,d}(y) = \tilde{J} \circ j_{a,d} = \left[\hat{J} \circ \phi + L\right] \circ j_{a,d} = \left(\hat{J} \circ i_a\right) \circ \left(i_a^{-1} \circ \phi \circ j_a\right)(d, y) + L \quad .$$

As in the proof of theorem 4.1, $\left(i_a^{-1} \circ \phi \circ j_a\right)(d, y) = (d \cdot k_a, y)$ for a 0-Čech cochain (k_a) which is determined by the cocycles (g_{ab}), (g'_{ab}) up to its value in a coset of the center of $\mathcal{D}$ in $\mathcal{D}$. Then the last equation gives

$$J'_{a,d} = J_{a,d \cdot k_a} + L \quad .$$

In summary, we have proven:

14.1 Theorem and Definition: Local moment map

Let Y be a connected, but not simply connected symplectic manifold with base point y, with $\mathcal{D} \equiv \pi_1(Y, y)$, and ω is the symplectic 2-form on Y. Let $\Phi : G \times Y \to Y$ be a symplectic left action of a connected Lie group G on Y with respect to ω. Let $\mathcal{V} = \{V_a \mid a \in A\}$ be a simply connected path-connected open cover of Y. Then

(A) for every $\mathcal{D}$-valued 1-Čech-cocycle (g_{ab}), $a, b \in A$, on $\mathcal{V}$, whose associated homomorphism $\rho : \mathcal{D} \to \mathcal{D}$ is an *inner* automorphism of $\mathcal{D}$, there exists a collection $(J_{a,d})$ of coalgebra-valued functions $J_{a,d} : V_a \to g^*$ for $a \in A$, $d \in \mathcal{D}$, such that

1.

$$\frac{\partial\Phi}{\partial G}A \vdash \omega + d\left\langle J_{a,d}, A\right\rangle = 0 \tag{77}$$

on V_a, and for all $d \in \mathcal{D}$;

2. let λ be a loop at y with $[\lambda] = d \in \pi_1(Y, y) \simeq \mathcal{D}$. Then

$$\langle J_{a,d}, A\rangle = \langle J_{a,e}, A\rangle - \int_\lambda \frac{\partial\Phi}{\partial G}A \vdash \omega \tag{78}$$

for all $A \in \hat{g}$, where e is the identity in $\mathcal{D}$.

3. the $J_{a,d}$ satisfy a *glueing condition*, expressed by

$$J_{a,d} = J_{b,d\cdot g_{ab}} \tag{79}$$

on $V_a \cap V_b \neq \emptyset$.

(B) Let (g'_{ab}) be another cocycle giving rise to a simply connected cover of Y, and let $\left(J'_{a,d}\right)$ be another collection of functions on $\mathcal{V}$ satisfying properties (A1–A3) with respect to (g'_{ab}) and the action Φ. Then there exists a Y-constant linear map $L : \hat{g} \to \mathbb{R}$ (i.e. $dL = 0$) and a $\mathcal{D}$-valued 0-Čech cochain $(k_a : V_a \to \mathcal{D})$ on $\mathcal{V}$ such that

$$J'_{a,d} = J_{a,d\cdot k_a} + L \tag{80}$$

for all $a \in A$, $d \in \mathcal{D}$. The 0-cochain (k_a) is determined by the cocycles (g_{ab}) and (g'_{ab}) as expressed in theorem 4.1.

(C) **Definition:** A collection $(g_{ab}; J_{a,d})$ satisfying properties (A1–A3) will be called a *local moment map* for the action Φ on the symplectic manifold (Y, ω).

15 Equivariance of moment maps

Usually, moment maps are introduced in a more restricted context. For example, conditions are imposed from the start so as to guarantee the existence of a uniquely determined single-valued globally defined moment map. Furthermore, it is often assumed that the first and second Chevalley-Eilenberg cohomology groups of the group G vanish, which then provides a sufficient condition for the moment map to transform as a G-morphism [2],[8]. Our approach will be slightly more general. The first generalization has been made above, allowing for moment maps to be only locally defined. The second one is, that we do not want to enforce the moment maps to behave as strict G-morphisms; rather, the deviation from transforming equivariantly is determined by a cocycle in a certain cohomology on $\hat{g}$, which in turn gives rise to a central extension of the

original Lie algebra $\hat{g}$, which is interesting in its own right and also physically relevant.

Let the conditions of theorem 14.1 be given. Reconstruct a simply connected cover $p : X \to Y$ from $\mathcal{V}$ and (g_{ab}) as in section 14, together with trivializations (i_a) such that $i_b^{-1} \circ i_a (d, y) = (d \cdot g_{ab}, y)$. Assume a lift $\hat{\Phi}_g$ exists. Assume that $p \circ \hat{\Phi}_g (x) \in V_b$, where $x = i_a (d, y)$; then there exists a unique $d' \in \mathcal{D}$ with $i_b^{-1} \circ \hat{\Phi}_g (x) = (d', \Phi_g (y))$, with $\Phi_g (y) \in V_b$. Here d' is a function of d, g, and y; its structure can be understood from the following consideration: Let λ be a path in G connecting e with g. Then $t \mapsto \Phi (\lambda (t), y)$ is a path in Y connecting y with $\Phi_g (y)$, which has a unique lift to $i_a (d, y)$, whose endpoint is just $\hat{\Phi}_g \circ i_a (d, y)$, by the definition of the lift $\hat{\Phi}$. The condition $\Phi_{y\#} \pi_1 (G, e) \subset \pi_1 (Y, y)$ guarantees that this construction is independent of the path λ. The interval $[0, 1]$ can be partitioned into subintervals $[a_{i-1}, a_i]$ so that $\Phi (\lambda (t), y) \in V_{a_i}$ for $t \in [a_{i-1}, a_i]$, and $\lambda (t_n) = g$. It follows that

$$i_{a_n}^{-1} \circ \hat{\Phi}_g (x) = \left(d \cdot g_{a_0 a_1} \cdots g_{a_{n-1} a_n}, \Phi_g (y) \right) \quad , \tag{81}$$

and hence $d' = d \cdot g_{a_0 a_1} \cdots g_{a_{n-1} a_n} \equiv d \cdot \psi_{a_n} (g, y)$. Since $\mathcal{D}$ is discrete, ψ_{a_n} is locally constant. Altogether we have shown that if $p \circ \hat{\Phi}_g (x) \in V_b$, then

$$i_b^{-1} \circ \hat{\Phi}_g (x) = (d \cdot \psi_b, \Phi_g (y)) \quad . \tag{82}$$

— Next we recall without proof the G-transformation behaviour for global moment maps (see [12]):

15.1 Theorem: G-transformation of global moment maps

Let (M, ω) be a simply connected symplectic manifold, let $\Phi : G \times M \to M$ be a symplectic left action of the Lie group G on M. Let J be a global moment map for the action Φ. Then

$$J \circ \Phi_g = Ad^* (g) \cdot J + \alpha (g) \quad , \tag{83}$$

where $\alpha : G \to g^*$ is a 1-cocycle in the g^*-valued cohomology on G as defined in section G, i.e. $\alpha \in Z^1 (G, g^*)$. This means that

$$(\delta\alpha) (g, h) = Ad^* (g) \cdot \alpha (h) - \alpha (gh) + \alpha (g) = 0 \tag{84}$$

for all $g, h \in G$. Thus, (83) says that J transforms equivariantly under G up to a cocycle in the g^*-valued cohomology.

— Now we generalize this result to local moment maps. We prove:

15.2 Theorem: G-transformation behaviour of local moment maps

Let Y be a connected symplectic manifold, with $\mathcal{D} \equiv \pi_1 (Y, y)$. Let $\Phi : G \times Y \to Y$ be a symplectic left action of a connected Lie group G on Y. Let

$\mathcal{V} = \{V_a \mid a \in A\}$ be a simply connected open cover of Y. Let (g_{ab}), $a, b \in A$, be a $\mathcal{D}$-valued 1-Čech-cocycle on $\mathcal{V}$ describing a simply connected $\mathcal{D}$-covering space of Y, and let the collection $(J_{a,d}; g_{ab})$ be the associated local moment map with respect to the action Φ.

Let $y \in Y$, and assume that $p \circ \Phi_g (y) \in V_b$. Then

$$J_{b,d\cdot\psi_b} \circ \Phi_g = Ad^* (g) \cdot J_{a,d} \; + \; \alpha (g) \quad , \tag{85}$$

where $\alpha : G \to g^*$ is a 1-cocycle in the g^*-valued cohomology on G as defined in appendix, section G, and ψ_g is defined in formula (82).

Proof :

Let the conditions of theorem 14.1 be given, and reconstruct a simply connected cover $p : X \to Y$ from $\mathcal{V}$ and (g_{ab}) as in section 14, together with trivializations (i_a) such that $i_b^{-1} \circ i_a (d, y) = (d \cdot g_{ab}, y)$. Assume a lift $\hat{\Phi}_g$ exists. By $\hat{J}$ we denote the global moment map on X with respect to $\hat{\Phi}$. This satisfies

$$\hat{J} \circ \hat{\Phi}_g = Ad^* (g) \cdot \hat{J} + \alpha (g) \quad , \tag{86}$$

according to (83). From (82) it follows that

$$i_b^{-1} \circ \hat{\Phi}_g \circ i_a (d, y) = (d \cdot \psi_b, \Phi_g (y)) \quad . \tag{87}$$

Using $J_{a,d} = \hat{J} \circ i_{a,d}$ in (86), (87) implies

$$\left[\hat{J} \circ i_b \right] \circ \left[i_b^{-1} \circ \hat{\Phi}_g \circ i_a \right] (d, y) = J_{b,d\cdot\psi_b} (y)$$

for the LHS, and hence the result (85). ∎

16 Non-simply connected coverings

In this section we study the relation between local moment maps on symplectic manifolds Z, Y where $q : Z \to Y$ is a covering of manifolds, but Z is not necessarily simply connected:

Let $\zeta \in Z$, $\eta \in Y$ be base points with $q (\zeta) = \eta$; let $\mathcal{D} \equiv \pi_1 (Y, \eta)$, and $\mathcal{H} \equiv \pi_1 (Z, \zeta)$. Let X be a universal covering manifold $p : X \to Y$ of Y such that Y is the orbit space $Y = X/\mathcal{D}$. Then X is also a universal cover of Z, and Z is isomorphic to the orbit space $X/\mathcal{H}$; for the sake of simplicity we ignore this isomorphism and identify $Z = X/\mathcal{H}$. There is a covering projection $r : X \to Z$, taking the base point $\xi \in X$ to $r (\xi) = \zeta$, and $p = q \circ r$. Since $q_\# \pi_1 (Z, \zeta)$ is an injective image of $\mathcal{H}$ in $\mathcal{D}$, we identify $\mathcal{H}$ with its image under $q_\#$, and thus can regard $\mathcal{H}$ as a subgroup of $\mathcal{D}$.

If ω is a symplectic form on Y, then $\Omega \equiv q^* \omega$ and $\hat{\Omega} \equiv p^* \omega$ are the natural symplectic forms on Z and X, respectively, and $\hat{\Omega} = r^* \Omega$.

Let the Lie group G act on Y from the left via $\Phi : G \times Y \to Y$. We assume that the lift $\hat{\Phi}$ of Φ to X exists, which is true if and only if $\Phi_{\eta \#} \pi_1 (G, e) = \{e\}$.

It is easy to show that in this case the lift $\tilde{\Phi}$ of Φ to Z exists; and furthermore, that the lift of $\tilde{\Phi}$ to X coincides with $\hat{\Phi}$.

We now introduce trivializations for $p : X \to Y$, and subsequently, construct preferred trivializations of the coverings $r : X \to Z$ and $q : Z \to Y$ based on this. Firstly, let $\mathcal{V}$ be a simply connected open cover of Y as above, and let $(i_a : \mathcal{D} \times V_a \to X)$ be a trivialization of the covering $p : X \to Y$. Given an element $d \in \mathcal{D}$, let $[d]$ denote the coset $[d] \equiv \mathcal{H} \cdot d$, where $\mathcal{H} \subset \mathcal{D}$ is regarded as a subgroup of $\mathcal{D}$, as above. For $V_a \in \mathcal{V}$ and $d \in \mathcal{D}$, define sets $U_{a,[d]} \equiv r \circ i_a (\{d\} \times V_a) \subset Z$. The sets $U_{a,[d]}$ are open and simply connected by construction, and their totality

$$\mathcal{U} \equiv \left\{U_{a,[d]} \mid a \in A\,;\ [d] \in \mathcal{D}/\mathcal{H}\right\} \tag{88}$$

covers Z, since the sets $i_a (d, V_a)$ cover X. The $U_{a,[d]}$ are just the connected components of the inverse image $q^{-1} (V_a) \subset Z$, hence we have $q \left(U_{a,[d]}\right) = V_a$ for all $[d] \in \mathcal{D}/\mathcal{H}$, and $\mathcal{U}$ is a simply connected open (countable) cover of Z. We define a trivialization $(k_a : \mathcal{D}/\mathcal{H} \times V_a \to Z)$ of the covering $q : Z \to Y$ by $k_a ([d] , y) \equiv r \circ i_a (d, y)$. Since r maps points $x \in X$ into orbits $r (x) = \hat{\Phi} (\mathcal{H}, x)$, this definition is independent of the representative d of $[d]$. Now we can construct trivializations of $r : X \to Z$ based on the cover $\mathcal{U}$ of Z; in particular, by specifying representatives d_0 of the various cosets $[d_0]$, we see that there exists a trivialization $\left(j_{a,[d_0]} : \mathcal{H} \times U_{a,[d_0]} \to X\right)$ such that

$$j_{a,[d_0]} (h, k_a ([d_0] , y)) = i_a (h \cdot d_0, y) \tag{89}$$

for all arguments.

As X is simply connected, the action $\hat{\Phi}$ has a global moment map $\hat{J}$ satisfying

$$\frac{\partial\hat{\Phi}}{\partial G} A \vdash p^*\omega + d \left\langle \hat{J}, A \right\rangle = 0 \quad . \tag{90}$$

Using the arguments in section 14 we find that

$$\frac{\partial\tilde{\Phi}}{\partial G} A \vdash q^*\omega + d \left\langle \hat{J} \circ j_{a,[d_0],h}, A \right\rangle = 0 \quad . \tag{91}$$

Hence, introducing the quantities $\tilde{J}_{a,[d_0],h} \equiv \hat{J} \circ j_{a,[d_0],h}$, and taking into account that the trivializations $\left(j_{a,[d_0]}\right)$ define an $\mathcal{H}$-valued 1-Čech cocycle $\left(\hat{g}_{a,[d_0]\,;\,b,\left[d_0'\right]}\right)$ by

$$j^{-1}_{b,\left[d_0'\right]} \circ j_{a,[d_0]} (h, z) = \left(h \cdot \hat{g}_{a,[d_0]\,;\,b,\left[d_0'\right]}, z\right) \quad , \tag{92}$$

we see that the collection $\left(\tilde{J}_{a,[d_0],h}\,;\, \hat{g}_{a,[d_0]\,;\,b,\left[d_0'\right]}\right)$ defines a local moment map for the action $\tilde{\Phi}$ with respect to $q^*\omega$. Similarly, the equation

$$\frac{\partial\Phi}{\partial G} A \vdash \omega + d \left\langle \hat{J} \circ i_{a,d}, A \right\rangle = 0 \tag{93}$$

shows that the collection $(J_{a,d}\,;\,g_{ab})$, where $J_{a,d} \equiv \hat{J} \circ i_{a,d}$, and (g_{ab}) is the $\mathcal{D}$-valued 1-Čech cocycle with respect to (i_a), is a local moment map for the action Φ with respect to ω. The relation between these two local moment maps is easily found using (89) to be

$$\tilde{J}_{a,[d_0],h} \circ k_{a,[d_0]} = J_{a,h\cdot d_0} \quad . \tag{94}$$

17 G-state spaces and moment maps

Finally, in this section we discuss our concept of a G-state space. This is an identification space based on a partitioning of a symplectic manifold into connected subsets, on each of which a given global moment map is constant. These connected subsets are then invariant under the Hamiltonian flow associated with every Hamiltonian h that commutes with the G-action $\hat{\Phi}$. This construction coincides with the first step in a Marsden-Weinstein reduction of the symplectic manifold. We first consider the case where the symplectic manifold is simply connected:

Let X be a simply connected symplectic manifold with symplectic 2-form Ω. Let $\hat{\Phi}$ be a symplectic action of a Lie group G on X. There exists a global moment map J associated with $\hat{\Phi}$. To every $x \in X$ we now assign the connected component $s\,(x)$ of $J^{-1}\,(J\,(x))$ that contains x; i.e., $s\,(x) \subset J^{-1}\,(J\,(x))$ is connected (in the induced topology), and $x \in s\,(x)$. The collection of all $s\,(x)$, as x ranges through X, is denoted as Σ_X. Then Σ_X is an identification space, where $s : X \to \Sigma_X$ is the identification map. We endow Σ_X with the quotient topology inherited from X. One can assume further technical conditions in order to guarantee that the sets $s\,(x)$ are presymplectic manifolds which give rise to reduced phase spaces; such a reduction is called Marsden-Weinstein reduction [2]. We do not make these assumptions here, since they are not necessary for our purposes.

By construction, the moment map J is constant on every connected component of $J^{-1}\,(J\,(x))$, and hence descends to the space Σ_X; i.e., there exists a unique map $\iota : \Sigma_X \to g^*$ satisfying $\iota \circ s = J$. Also, every diffeomorphism $\hat{\Phi}_g$ maps connected components of $J^{-1}\,(J\,(x))$ into connected components; this follows from formula (83). Hence, $\hat{\Phi}$ descends to an action $\hat{\phi} : G \times \Sigma_X \to \Sigma_X$, satisfying

$$s \circ \hat{\Phi}_g = \hat{\phi}_g \circ s \quad , \tag{95}$$

which gives rise to an analogue of formula (83) on Σ_X,

$$\iota \circ \hat{\phi}_g = Ad^*\,(g) \cdot \iota + \alpha\,(g) \quad . \tag{96}$$

In this construction, we have identified all states $x \in X$ which are mapped into the same value under the moment map, and which can be connected by a path on which the moment map is constant. Σ_X is a G-space with action $\hat{\phi}$. There is a semi-equivariant map ι from Σ_X to the G-space g^*. Furthermore, the connected components $s\,(x)$ are preserved by any Hamiltonian that commutes with G. This discussion can be summarized in the

17.1 Theorem and definition

Let X be a simply connected symplectic manifold, let $\hat{\Phi}$ be a symplectic action of a connected Lie group G with Lie algebra $\hat{g}$ on X, let J be a global moment map associated with $\hat{\Phi}$. Then

(A) there exists a space Σ_X with a G-action $\hat{\phi}$, a projection $s : X \to \Sigma_X$, and a semi-equivariant map $\iota : \Sigma_X \to g^*$ satisfying $\iota \circ s = J$ such that (96) holds.

(B) If h is any Hamiltonian on X satisfying the Poisson-bracket relations $\{h, \langle J, A\rangle\} = 0$ for all $A \in \hat{g}$, then the associated Hamiltonian flow $f_t(x)$ preserves the sets $s^{-1}(\sigma)$, $\sigma \in \Sigma$. In other words,

$$J \circ f_t(x) = J(x) \tag{97}$$

for all $x \in s^{-1}(\sigma)$ and $t \in \mathbb{R}$.

(C) Σ_X will be called a *G-state space* for the pair $\left(X, \hat{\Phi}\right)$.

18 The splitting of multiplets

Now we turn to investigate the relation of the objects defined above to a similar construction on a non-simply connected manifold Y, where $p : X \to Y$ is a universal covering of Y, and p is a local symplectomorphism of symplectic forms ω on Y and $\Omega \equiv p^*\omega$ on X. The first thing to observe is that the diffeomorphisms γ of the deck transformation group $\mathcal{D}$ of the covering descend to Σ_X: To see this, let λ be a path lying entirely in one of the connected components $s^{-1}(\sigma) \subset J^{-1}(J(x))$ [Here we assume that connectedness implies path-connectedness]. If $\dot{\lambda}$ denotes its tangent, we have

$$\frac{d}{dt}\langle J \circ \gamma \circ \lambda, A\rangle = \dot{\lambda} \vdash \gamma^* d\langle J, A\rangle = -\Omega\left(\frac{\partial\hat{\Phi}}{\partial G}A, \gamma_*\dot{\lambda}\right) =$$

$$= -(\gamma^*\Omega)\left(\gamma_*^{-1}\frac{\partial\hat{\Phi}}{\partial G}A, \dot{\lambda}\right) = -\Omega\left(\frac{\partial\hat{\Phi}}{\partial G}A, \dot{\lambda}\right) = \frac{d}{dt}\langle J \circ \lambda, A\rangle = 0 \quad .$$

Here, the last equation follows from the fact that λ lies in a connected component of $J^{-1}(J(x))$; furthermore, we have used that Ω is $\mathcal{D}$-invariant, and the action $\hat{\Phi}$ commutes with $\mathcal{D}$, since G is connected (this follows from theorem 10.2). But this result says that J is constant on the γ-image of every connected component $J^{-1}(J(x))$; since this image is connected itself, it must lie in one of the connected components of $J^{-1}(J \circ \gamma(x))$. As γ is invertible, it follows that γ maps connected components onto connected components, and hence descends to a map $\bar{\gamma} : \Sigma_X \to \Sigma_X$ such that

$$\bar{\gamma} \circ s = s \circ \gamma \quad . \tag{98}$$

Thus, we have a well-defined action of $\mathcal{D}$ on Σ_X. We now construct a space Σ_Y analogous to Σ_X: Define Σ_Y as the quotient $\Sigma_X/\mathcal{D}$, with projection $q : \Sigma_X \to \Sigma_Y$. We note that this is not a covering space in general, since the action of $\mathcal{D}$ on Σ_X need not necessarily be free; for example, if γ maps one of the connected components $s^{-1}(\sigma)$ onto itself, then $\bar{\gamma}$ has a fixpoint on Σ_X. However, formula (98) implies that the map s descends to the quotient $\Sigma_Y = \Sigma_X/\mathcal{D}$, which means that there exists a unique map $\bar{s} : Y \to \Sigma_Y$ such that

$$\bar{s} \circ p = q \circ s \quad . \tag{99}$$

Using (98) and (99) it is easy to see that the action Φ of G on Y preserves the G-states on Y, i.e. the images $p \circ s^{-1}(\sigma) = \bar{s}^{-1} \circ q(\sigma)$ of the connected components of $J^{-1}(J(x))$, where $x \in s^{-1}(\sigma)$, under p: For, let $y \in p \circ s^{-1}(\sigma)$, then there exists an $x \in s^{-1}(\sigma)$ with $y = p(x)$. Then

$$\Phi_g(y) = \Phi_g \circ p(x) = p \circ \hat{\Phi}_g(x) \in p \circ \hat{\Phi}_g \circ s^{-1}(\sigma) = p \circ s^{-1} \circ \hat{\phi}_g(\sigma) = \bar{s}^{-1} \circ q \circ \hat{\phi}_g(\sigma) \quad ,$$

which implies that $\bar{s} \circ \Phi_g(y) \in q \circ \hat{\phi}_g(\sigma)$ for all y in $\bar{s}^{-1}(q(\sigma))$. Hence, Φ descends to an action $\phi : G \times \Sigma_Y \to \Sigma_Y$ with

$$\phi_g \circ \bar{s} = \bar{s} \circ \Phi_g \quad , \tag{100}$$

which is the analogue of (95).

The orbits $\hat{\phi}_G(\sigma)$, $\sigma \in \Sigma_X$, and $\phi_G(\tau)$, $\tau \in \Sigma_Y$, are the classical analogue of carrier spaces of irreducible G-representations in the quantum context [2]. However, for every G-state $\tau \in \Sigma_Y$ there exists a collection $q^{-1}(\tau)$ of G-states in Σ_X which are identified under q. The elements in the collection $q^{-1}(\tau)$ are labelled by the elements d of the fundamental group $\mathcal{D} \simeq \pi_1(Y, y)$. We call this phenomenon the "splitting of (classical) multiplets" on account of the multiple-connectedness of the background Y. It is basically a consequence of the fact that the group G, when lifted to the covering space X, is extended to a group $\tilde{G}$ by the deck transformation group $\mathcal{D}$, whose group law in the case under consideration is determined by formula (50) in theorem 10.2.

A Appendix

Here we compile the cohomologies that are used in this work. References are [3],[11],[12].

B q-form-valued cohomology on the deck transformation group

Let $p : X \to Y$ be a covering of manifolds, where X is simply connected. The deck transformation group of the covering is $\mathcal{D}$. Our analysis is based on formula (3),

$$\gamma^* \eta = \eta + d\chi(\gamma) \quad , \tag{101}$$

where η is the potential for a q-form $d\eta$ which is the pull-back of a closed q-form ω on Y, i.e. $d\eta = p^*\omega$. Then $d\eta$ is $\mathcal{D}$-invariant, as follows from proposition 3.2, but η is not, as follows from the last equation. In particular this means that for $\chi \neq 0$, η is not the pull-back under p^* of a form on Y. Now (101) defines a cochain in a cohomology on $\mathcal{D}$ defined as follows (our notation conventions are those of [12]): An n-cochain α_n is a map $\alpha_n : \mathcal{D}^n \to \Lambda^*(X)$, where $\Lambda^*(X) = \bigoplus_{q\geq 0} \Lambda^q(X)$ denotes the ring of differential forms on X. The deck transformation group $\mathcal{D}$ acts via pull-back of elements γ on forms: $\mathcal{D} \times \Lambda^*(X) \ni (\gamma, \alpha) \mapsto \gamma^*\alpha$. This is a **right** action, in contrast to the the cohomologies to be discussed below. A zero-cochain α_0 is an element of $\Lambda^*(X)$. The coboundary operator δ in this cohomology is defined to act on 0-, 1-, 2-cochains according to

$$\begin{aligned} (\delta\alpha_0)(\gamma) &= \gamma^*\alpha_0 - \alpha_0 \quad , \\ (\delta\alpha_1)(\gamma_1,\gamma_2) &= \gamma_2^*\alpha_1(\gamma_1) - \alpha_1(\gamma_1\gamma_2) + \alpha_1(\gamma_2) \quad , \\ (\delta\alpha_2)(\gamma_1,\gamma_2,\gamma_3) &= \gamma_3^*\alpha_2(\gamma_1,\gamma_2) + \alpha_2(\gamma_1\gamma_2,\gamma_3) - \\ &\quad -\alpha_2(\gamma_1,\gamma_2\gamma_3) - \alpha_2(\gamma_2,\gamma_3) \quad , \end{aligned} \tag{102}$$

and δ is nilpotent, $\delta \circ \delta = 0$, as usual. Now consider the 0-cochain η in equation (101). With the help of (102), equation (101) can be expressed as

$$\delta\eta = d\chi \quad , \tag{103}$$

where it is understood that χ is a function of arguments in the set $\mathcal{D} \times X$. The commutativity $[\delta, d] = 0$ and the nilpotency $\delta^2 = 0$ and $d^2 = 0$ of the coboundary operators give rise to a chain of equations similar to (103): Applying δ to (103) gives

$$d(\delta\chi) = 0 \quad . \tag{104}$$

Since X is simply connected

$$\delta\chi = d\chi' \tag{105}$$

for some χ'. Now the process can be repeated with the last equation, etc.

C Čech cohomology

The definitions in this section are based on [3],[11].

Let Y be a topological space and $\mathcal{D}$ be a discrete group. Let $\mathcal{V} \equiv \{V_a \mid a \in A\}$ be an open cover of Y such that every V_a is admissible. [If Y is a manifold, we can assume that A is countable, and every V_a is simply connected.] A function $g : Y \to \mathcal{D}$ on the topological space Y is called *locally constant* if every point $y \in Y$ possesses a neighbourhood V on which the restriction of g is constant.

Let $\mathcal{S}^0$ denote the sum $\mathcal{S}^0 \equiv \bigsqcup_{a\in A} V_a$; let $\mathcal{S}^1$ denote the sum $\mathcal{S}^1 \equiv \bigsqcup_{a,b} V_a \cap V_b$, for all a, b for which $V_a \cap V_b \neq \emptyset$, allowing for $a = b$. We denote the images of V_a, $V_a \cap V_b$, ..., under the associated injections simply by (a), (a, b), etc. [We

recall that the set underlying a sum $\Sigma = B \sqcup C$ is the disjoint union of B and C. Furthermore, if $i : B \to \Sigma$, $j : C \to \Sigma$ are the injections, then $i(B)$ and $j(C)$ are both open and closed in Σ, which means that Σ is disconnected. Hence if each V_a is connected in Y, then a locally constant function on $\mathcal{S}^0$ is constant on the images of all V_a under the appropriate injection; similarly, a locally constant function on $\mathcal{S}^1$ is constant on the images of $V_a \cap V_b$ under injection. Therefore in this case, a locally constant function on $\mathcal{S}^0, \mathcal{S}^1$ is constant on all (a), and (a, b), respectively]. A locally constant function $f_0 : \mathcal{S}^0 \to \mathcal{D}$ is called a 0-Čech-cochain (with respect to $\mathcal{V}$). A locally constant function $f_1 : \mathcal{S}^1 \to \mathcal{D}$ is called a 1-Čech-cochain (with respect to $\mathcal{V}$). A 1-Čech-cochain f_1 is called a 1-*Čech-cocycle* if

(Coc1) $f_1|(a, a) = e$, where e is the identity in $\mathcal{D}$,

(Coc2) $f_1|(b, a) = f_1^{-1}\big|(a, b)$, f_1^{-1} denoting the inverse of f_1 in $\mathcal{D}$;

and for all V_a, V_b, V_c for which $V_a \cap V_b \cap V_c \neq \emptyset$ it is true that

(Coc3) $f_1|(a, c) = [f_1|(a, b)] \cdot [f_1|(b, c)]$,

where (a, b), (b, a), (a, c), etc., are to be regarded as disjoint subsets of $\mathcal{S}^1$.

Two 1-Čech-cocycles f, f' are said to be *cohomologous* if there exists a 0-Čech-cochain h such that

(Coh) $f'|(a, b) = \left[h^{-1}\big|(a)\right] \cdot [f|(a, b)] \cdot [h|(b)]$.

The property of being cohomologous defines an equivalence relation amongst all 1-Čech-cocycles with respect to $\mathcal{V}$; the equivalence classes are called *first Čech cohomology classes on* $\mathcal{V}$ with coefficients in $\mathcal{D}$. The set of these classes is denoted as $H^1(\mathcal{V}; \mathcal{D})$.

For $n > 1$, the n-th cohomology class is described more readily when $\mathcal{D}$ is Abelian. Assuming this, an n-*Čech-cochain* f_n with coefficients in $\mathcal{D}$ is a locally constant map $f_n : \mathcal{S}^n \to \mathcal{D}$, where $\mathcal{S}^n$ is the topological sum $\mathcal{S}^n \equiv \bigsqcup_{a_0,\dots,a_n} V_{a_0} \cap V_{a_1} \cdots \cap V_{a_n}$, for all $a_0, \dots, a_n$ for which $V_{a_0} \cap V_{a_1} \cdots \cap V_{a_n} \neq \emptyset$. The coboundary operator δ sends n-cochains f_n to $(n+1)$-cochains δf_n defined by [11]

$$\delta f_n|(a_0, \dots, a_{n+1}) = \sum_{k=0}^{n+1} (-1)^k \cdot f_n|(a_0, \dots, \widehat{a_k}, \dots, a_{n+1}) \quad , \tag{106}$$

where $\widehat{a_k}$ means that this argument has to be omitted, and $-f_n$ denotes the inverse of f_n in the Abelian group $\mathcal{D}$. δ is nilpotent, $\delta_{n+1} \circ \delta_n = 0$. As usual, n-cocycles are elements in $\ker \delta_n$, n-coboundaries are elements in $\operatorname{im} \delta_{n-1}$, and the n-th Čech cohomology group on $\mathcal{V}$ is the quotient $H^n(\mathcal{V}; \mathcal{D}) = \ker \delta_n / \operatorname{im} \delta_{n-1}$. Obviously, statements (Coc1-Coc3) and (Coh) above generalize this pattern to non-Abelian groups $\mathcal{D}$.

D $\mathcal{D}$-coverings

This and the next section are mainly based on [3].

Let $\mathcal{D}$ denote a discrete group. Let X be a topological space on which $\mathcal{D}$ acts properly discontinuously and freely. Let $p : X \to X/\mathcal{D}$ denote the projection onto the space of orbits, endowed with the quotient (final) topology. Then p is a covering map.

Generally, if a covering $p : X \to Y$ arises in this way from a properly discontinuous and free action $\mathcal{D}$ on X, we call the covering a *$\mathcal{D}$-covering*. Given two coverings $p : X \to Y$ and $p' : X' \to Y'$ of topological spaces, a homeomorphism $\phi : X \to X'$ is called *isomorphism of coverings* if ϕ is fibre-preserving, $p' \circ \phi = p$. An *isomorphism of $\mathcal{D}$-coverings* is an isomorphism of coverings that commutes with the actions of $\mathcal{D}$, i.e. $\phi\,(d \cdot x) = d \cdot \phi\,(x)$; in this case we also say that ϕ is *$\mathcal{D}$-equivariant*, and we say that the spaces involved are *$\mathcal{D}$-isomorphic*. The *trivial $\mathcal{D}$-covering* of Y is the Cartesian product $\mathcal{D} \times Y$ together with projection onto the second factor as covering map, and $\mathcal{D}$ acts on $(d, y) \in \mathcal{D} \times Y$ by left multiplication on the first factor, $(d', (d, y)) \mapsto (d'd, y)$. An isomorphism $i : \mathcal{D} \times Y \to X$ of the trivial $\mathcal{D}$-covering onto a $\mathcal{D}$-covering X is called a *trivialization of X*.

Let $p : X \to X/\mathcal{D} = Y$ be a $\mathcal{D}$-covering. Let $V \subset Y$ be an admissible connected open set in Y. A choice of a connected component $U \subset p^{-1}(V)$ defines a *trivialization* $i : \mathcal{D} \times V \to p^{-1}(V)$ of the $\mathcal{D}$-covering $p : p^{-1}(V) \to V$ as follows: For $(d, y) \in \mathcal{D} \times V$, let $i\,(d, y) \equiv d \cdot (p|\,U)^{-1}\,(y)$. Then i is evidently a bijection and hence a homeomorphism; furthermore it is fibre-preserving, since projection onto the second factor of (d, y) yields the same as $p \circ i$ applied to (d, y); and it is $\mathcal{D}$-equivariant by definition. This says that a $\mathcal{D}$-covering is trivial over each admissible neighbourhood $V \subset Y$. A different choice U' of connected components in $p^{-1}(V)$ defines a trivialization $i' : \mathcal{D} \times V \to p^{-1}(V)$ with $i'^{-1} \circ i\,(d, y) = (d \cdot g\,(y)\,, y)$, where $g : V \to \mathcal{D}$ is continuous, and hence constant on every connected subset of V, since $\mathcal{D}$ is discrete.

E Čech cohomology and the glueing of $\mathcal{D}$-coverings

Let $p : X \to X/\mathcal{D} = Y$ be a $\mathcal{D}$-covering; let $\mathcal{V}$ be an open cover of Y by admissible subsets $V \subset Y$. For sufficiently simple spaces such as manifolds it can be assumed that every $V \in \mathcal{V}$ is simply connected in Y and path-connected. As explained in the last section, a choice of connected component $U_a \subset p^{-1}(V_a)$ in the inverse image of every V_a, $a \in A$, gives rise to a set of local trivializations $i_a : \mathcal{D} \times V_a \to p^{-1}(V_a)$, which, in turn, define a collection (g_{ab}) of transition functions $g_{ab} : V_a \cap V_b \to \mathcal{D}$, $i_b^{-1} \circ i_a\,(d, y) = (d \cdot g_{ab}\,(y)\,, y)$. If all V_a are connected, the transition functions g_{ab} are constant due to continuity. It is easily seen that the (g_{ab}) satisfy

(Trans1) $g_{aa} = e$,

(Trans2) $g_{ba} = g_{ab}^{-1}$,

(Trans3) $g_{ac} = g_{ab} \cdot g_{bc}$,

the last equation following from $i_a^{-1} i_c i_c^{-1} i_b i_b^{-1} i_a = id$, whenever $V_a \cap V_b \cap V_c \neq \emptyset$. Comparison with (Coc1-Coc3) in section C shows that the collection (g_{ab}) defines a 1-Čech-cocycle on $\mathcal{V}$. Now suppose we choose different trivializations i'_a. Then the trivializations are related by $i'^{-1}_a \circ i_a (d, y) = (d \cdot h_a (y), y)$, with a collection (h_a) of locally constant functions $h_a : V_a \to \mathcal{D}$, which defines a 0-Čech-cochain on $\mathcal{V}$, as explained in section C. The transition functions (g'_{ab}) associated with (i'_a) are defined by $i'^{-1}_b \circ i'_a (d, y) = (d \cdot g'_{ab}, y)$; on the other hand, from the definition of (h_a), we find that $i'_a (d, y) = i_a \left(d \cdot h_a^{-1}, y\right)$, which implies that $i'^{-1}_b \circ i'_a (d, y) = \left(d \cdot h_a^{-1} g_{ab} h_b, y\right)$. Thus,

$$g'_{ab} = h_a^{-1} \cdot g_{ab} \cdot h_b \quad . \tag{107}$$

Statement (Coh) in section C now shows that the cocycles (g_{ab}) and (g'_{ab}) are cohomologous. This means that a $\mathcal{D}$-covering $p : X \to X/\mathcal{D} = Y$ determines a unique Čech cohomology class in $H^1 (\mathcal{V}; \mathcal{D})$. – Furthermore, let $q : Z \to Z/\mathcal{D} = Y$ be another $\mathcal{D}$-covering of Y such that there exists an isomorphism $\phi : X \to Z$ of $\mathcal{D}$-coverings. Assume that $i_a : \mathcal{D} \times V_a \to p^{-1} (V_a) \subset X$ is the trivialization over V_a in the $\mathcal{D}$-covering $p : X \to Y$. Then $(d, y) \mapsto \phi \circ i_a (d, y)$ is a trivialization over V_a in the covering $q : Z \to Y$ with transition functions $g'_{ab} = g_{ab}$. Any other trivialization of $q : Z \to Y$ gives a cohomologous cocycle. Thus, we have found that $\mathcal{D}$-coverings which are $\mathcal{D}$-isomorphic define a unique Čech cohomology class in $H^1 (\mathcal{V}; \mathcal{D})$.

– Conversely, we want to show that a cohomology class represented by (g_{ab}) defines a $\mathcal{D}$-covering of Y up to $\mathcal{D}$-isomorphisms. Let $p : X \to Y = X/\mathcal{D}$, $p' : X' \to Y = X'/\mathcal{D}$ be two $\mathcal{D}$-coverings with trivializations $i_a : \mathcal{D} \times V_a \to p^{-1} (V_a)$, $i'_a : \mathcal{D} \times V_a \to p'^{-1} (V_a)$ and associated cocycles (g_{ab}), (g'_{ab}) defined by $i_b^{-1} \circ i_a (d, y) = (d \cdot g_{ab}, y)$, $i'^{-1}_b \circ i'_a (d, y) = (d \cdot g'_{ab}, y)$, such that $g'_{ab} = h_a^{-1} \cdot g_{ab} \cdot h_b$, where $(h_a : V_a \to \mathcal{D})$ is a 0-Čech-cochain on $\mathcal{V}$. It follows that $i'_a (d \cdot h_a, y) = i'_b (d \cdot g_{ab} \cdot h_b, y)$. On the sets $\mathcal{D} \times V_a$ we now define a collection of functions $k_a : \mathcal{D} \times V_a \to X'$ determined by $k_a (d, y) = i'_a (d \cdot h_a, y)$. By construction, all k_a are continuous. Furthermore, if $(d, y) \in \mathcal{D} \times V_a$ and $(d', y') \in \mathcal{D} \times V_b$ are identified under the trivializations i_a, i_b, so that $i_a (d, y) = i_b (d', y')$, then $k_a (d, y)$ and $k_b (d', y')$ coincide; for, in this case, we must have $(d', y') = (d \cdot g_{ab}, y)$, and hence

$$k_b (d', y') = i'_b (d' \cdot h_b, y') = i'_b (d \cdot g_{ab} \cdot h_b, y) = i'_b (dh_a \cdot g'_{ab}, y) =$$

$$= i'_a (d \cdot h_a, y) = k_a (d, y) \quad .$$

Now the set of trivializations $(i_a : \mathcal{D} \times V_a \to X)$ as defined here can be regarded as a collection of identification maps (i_a) on the sets $\mathcal{D} \times V_a$, where $\mathcal{D}$ has the discrete topology, and the V_a have the topology induced from Y. The identification space X has the ϕ-universal property [7] that for every topological space X' and any collection of continuous functions $(k_a : \mathcal{D} \times V_a \to X')$ which coincide on elements (d, y), (d', y') which are identified in the identification space

$(i_a : \mathcal{D} \times V_a \to X)$, there exists a unique continuous map $\psi : X \to X'$ such that $\psi \circ i_a = k_a$. Since the arguments leading to this result can be reversed, it follows that ψ has a continuous inverse, and hence is a homeomorphism. Locally, we have $i_a'^{-1} \circ \psi \circ i_a (d, y) = (d \cdot h_a, y)$, which says that ψ is fibre-preserving, and hence is a covering isomorphism. The same formula shows that ψ is $\mathcal{D}$-equivariant. Altogether, therefore, ψ is an isomorphism of $\mathcal{D}$-coverings.

The last two paragraphs therefore prove that there is a 1–1 correspondence between $\mathcal{D}$-coverings of Y which are $\mathcal{D}$-isomorphic, and cohomology classes in the first Čech cohomology group $H^1(\mathcal{V}; \mathcal{D})$ on $\mathcal{V}$ with values in $\mathcal{D}$.

F Deck-transformation-valued cohomology on G

Consider the scenario of theorem 10.2; there it was shown that when the action $\phi : G \times Y \to Y$ of the Lie group G is lifted to a smooth map $\hat{\phi} : G \times X \to X$ satisfying (L1,L2), then the set $\left\{ \hat{\phi}_g \mid g \in G \right\}$ usually no longer closes into a group, but is extended to a larger group $\tilde{G}$ which contains $\mathcal{D}$ as a normal subgroup such that $\tilde{G}/\mathcal{D} = G$. The deviation from closure was measured by the map

$$\Gamma : G \times G \to \mathcal{D} \quad , \quad (g, h) \mapsto \Gamma(g, h) \equiv \hat{\phi}_g \hat{\phi}_h \widehat{\phi_{gh}}^{-1} \quad , \tag{108}$$

see (43). Furthermore, we have a map

$$b(g) : \mathcal{D} \to \mathcal{D} \quad , \quad \gamma \mapsto b(g)\gamma \equiv \hat{\phi}_g \circ \gamma \circ \hat{\phi}_g^{-1} \quad . \tag{109}$$

In the discussion following formula (42) it was pointed out that $b : G \to Aut(\mathcal{D})$ usually is not a representation; here we show that if $\mathcal{D}$ is Abelian, then b **is** a representation, and hence defines a left action

$$G \times \mathcal{D} \to \mathcal{D} \quad , \quad (g, \gamma) \mapsto b(g)\gamma \quad , \tag{110}$$

of G on $\mathcal{D}$. To see this, consider the expression

$$b(gg')\gamma = \hat{\phi}_{gg'} \gamma \hat{\phi}_{gg'}^{-1} \quad ;$$

using (108,109) this becomes

$$\hat{\phi}_{gg'} \gamma \hat{\phi}_{gg'}^{-1} = \Gamma^{-1}(g, g') \left[b(g) \circ b(g') \gamma \right] \Gamma(g, g') \quad ;$$

but the expression $b(g) \circ b(g')\gamma$ in square brackets is an element of $\mathcal{D}$, as are the Γ's. Hence, since $\mathcal{D}$ is Abelian, the last expression is

$$\Gamma^{-1}(g, g') \left[b(g) \circ b(g') \gamma \right] \Gamma(g, g') = b(g) \circ b(g') \gamma \quad ,$$

which proves

$$b(gg') = b(g) \circ b(g') \quad . \tag{111}$$

In the sequel we use an additive notation for the group law in $\mathcal{D}$; i.e. $(\gamma, \gamma') \mapsto \gamma + \gamma' \in \mathcal{D}$. We now introduce a $\mathcal{D}$-valued cohomology on G as follows: n-cochains α_n are maps $G^n \to \mathcal{D}$; 0-cochains are elements of $\mathcal{D}$. The coboundary operator δ is defined to act on 0-, 1-, 2-cochains according to

$$\begin{array}{rcl} (\delta\alpha_0)(g) & = & b(g)\,\alpha_0 - \alpha_0 \quad , \\ (\delta\alpha_1)(g,h) & = & b(g)\,\alpha_1(h) - \alpha_1(gh) + \alpha_1(g) \quad , \\ (\delta\alpha_2)(g,h,k) & = & b(g)\,\alpha_2(h,k) + \alpha_2(g,hk) - \\ & & -\alpha_2(gh,k) - \alpha_2(g,h) \quad . \end{array} \tag{112}$$

This is well-defined, since (111) says that b is now an action. We denote the sets of n-cochains, -cocycles, -coboundaries, and n-cohomology groups by $C^n(G,\mathcal{D})$, $Z^n(G,\mathcal{D})$, $B^n(G,\mathcal{D})$, and $H^n(G,\mathcal{D}) = Z^n(G,\mathcal{D})/B^n(G,\mathcal{D})$.

— Another cohomology on G and $\hat{g}$ that occurs in studying moment maps is the

G g^*-valued cohomology on G

n-cochains are smooth maps $\alpha_n : G^n \to g^*$. G acts on g^* via the coadjoint representation Ad^* of G on g^*; this is a left action, see the beginning of the appendix. The set of all g^*-valued n-cochains is denoted by $C^n(G,g^*)$. The coboundary operator $\delta : C^n \to C^{n+1}$ acts on 0-, 1-, 2-cochains α_0, α_1, α_2 according to

$$\begin{array}{rcl} (\delta\alpha_0)(g) & = & Ad^*(g)\,\alpha_0 - \alpha_0 \quad , \\ (\delta\alpha_1)(g,h) & = & Ad^*(g)\,\alpha_1(h) - \alpha_1(gh) + \alpha_1(g) \quad , \\ (\delta\alpha_2)(g,h,k) & = & Ad^*(g)\,\alpha_2(h,k) + \alpha_2(g,hk) - \\ & & -\alpha_2(gh,k) - \alpha_2(g,h) \quad , \end{array} \tag{113}$$

etc.. The set of all n-cocycles is denoted by $Z^n(G,g^*)$, the set of all n-coboundaries is denoted as $B^n(G,g^*)$. The n-th cohomology group of G with values in g^* is the quotient $H^n(G,g^*) = Z^n(G,g^*)/B^n(G,g^*)$.

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